Consider the one-dimensional heat equation

\[
\frac{u_t}{u_{xx}}(x, t) = 0 < x < L, \quad 0 < t \leq T; \quad u(0, t) = u(L, t) = 0; \quad u(x, 0) = f(x),
\]

The idea is to reduce this PDE to a system of ODEs by discretizing the equation in space, and then apply a suitable numerical method to the resulting system of ODEs.

Denote by \( \Delta x = L/N \) the step size in space and approximate \( u_{xx}(n\Delta x, t) \) for every \( t \in [0, T] \) using central differences:

\[
u_{xx}(n\Delta x, t) \approx \frac{1}{(\Delta x)^2} \left( u((n+1)\Delta x, t) - 2u(n\Delta x, t) + u((n-1)\Delta x, t) \right).
\]

Define the column vector \( U(t) = \left( U_1(t), \ldots, U_{N-1}(t) \right)^T \) as the solution of the system of equations

\[
\frac{dU_n(t)}{dt} = \frac{a}{(\Delta x)^2} \left( U_{n+1}(t) - 2U_n(t) + U_{n-1}(t) \right), \quad n = 1, \ldots, N-1, \quad 0 < t \leq T,
\]

with initial condition \( U_n(0) = f(n\Delta x) \), and set \( U_0(t) = U_N(t) = 0 \) for all \( t \). By (2), it is natural to consider \( U_n(t) \) as an approximation of \( u(n\Delta x, t) \). Note that, from the definition of \( U \) and the boundary conditions for \( u \), we have \( u(0, t) = 0 = U_0(t) \) and \( u(N\Delta x, t) = u(L, t) = 0 = U_N(t) \) for all \( t \).

In the matrix form, (3) becomes

\[
\frac{dU(t)}{dt} = \frac{a}{(\Delta x)^2} A_{N-1}^{[-2,1]} U(t), \quad 0 < t \leq T,
\]

where \( A_{N-1}^{[-2,1]} \) is a tri-diagonal square matrix of the size \( (N-1) \times (N-1) \):

\[
A_{N-1}^{[-2,1]} = \begin{pmatrix}
-2 & 1 & 0 & \cdots & 0 \\
1 & -2 & 1 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 1 & \\
0 & \cdots & 0 & 1 & -2
\end{pmatrix}
\]

Next, we discretize time by introducing a uniform grid with step \( \Delta t = T/M \). By (4),

\[
U((m+1)\Delta t) = U(m\Delta t) + \frac{a}{(\Delta x)^2} \int_{m\Delta t}^{(m+1)\Delta t} A_{N-1}^{[-2,1]} U(s) ds.
\]

We approximate the integral on the right-hand side of (6) by the trapezoidal rule:

\[
U((m+1)\Delta t) \approx U(m\Delta t) + \frac{a\Delta t}{2(\Delta x)^2} \left( A_{N-1}^{[-2,1]} U(m\Delta t) + A_{N-1}^{[-2,1]} U((m+1)\Delta t) \right).
\]

To proceed, let us introduce the notation

\[
r = \frac{a\Delta t}{2(\Delta x)^2}.
\]
Now define the sequence of vectors $\tilde{u}(m)$, $m = 0, \ldots, M$, by $\tilde{u}_n(0) = f(n\Delta x)$ (f is the initial condition from (1)), and

\begin{equation}
\tilde{u}(m + 1) = \tilde{u}(m) + r \left( A_{N-1}^{-2,1}[\tilde{u}(m)] + A_{N-1}^{[-2,1]}\tilde{u}(m + 1) \right).
\end{equation}

According to (7), $\tilde{u}(m)$ can be considered an approximation of $U(m^2t)$, and therefore $\tilde{u}_n(m)$ is an approximation of $u(n\Delta x, m^2t)$. Similar to the vector $U$, the vector $\tilde{u}(m)$ in (8) has $N - 1$ components $\tilde{u}_1(m), \ldots, \tilde{u}_{N-1}(m)$. The resulting approximation of the solution of equation (2) at time $m^2t$ and points $n\Delta x$, $n = 0, \ldots, N$, is $(0, \tilde{u}_1(m), \ldots, \tilde{u}_{N-1}(m), 0)$.

To compute $\tilde{u}(m)$, note that (8) can be written as follows:

\begin{equation}
A_{N-1}^{[1+2r, -r]}\tilde{u}(m + 1) = A_{N-1}^{[1-2r, r]}\tilde{u}(m),
\end{equation}

where, similar to (5),

\begin{equation}
A_{N-1}^{[1+2r, -r]} = \begin{pmatrix}
1 + 2r & -r & 0 & \cdots & 0 \\
-r & 1 + 2r & -r & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & -r \\
0 & \cdots & \cdots & 0 & 1 + 2r \\
\end{pmatrix},
A_{N-1}^{[1-2r, r]} = \begin{pmatrix}
1 - 2r & r & 0 & \cdots & 0 \\
r & 1 - 2r & r & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & r \\
0 & \cdots & \cdots & 0 & 1 - 2r \\
\end{pmatrix}
\end{equation}

(VERIFY THIS!!!)

and therefore

\begin{equation}
\tilde{u}(m + 1) = \left( A_{N-1}^{[1+2r, -r]} \right)^{-1} A_{N-1}^{[1-2r, r]}\tilde{u}(m)
\end{equation}

The method of computing an approximation of the solution of (1) according to (11) is called the Crank-Nicolson scheme. It was proposed in 1947 by the British physicists JOHN CRANK (b. 1916) and PHYLLIS NICOLSON (1917–1968).