**Definition:** A stochastic process $\{X(t) : t \geq 0\}$ is a Gaussian process if for every $n \geq 1$ and all times $t_1, \ldots, t_n$, the variables $(X(t_1), X(t_2), \ldots, X(t_n))$ have a multivariate Gaussian distribution.

**Observe:** A Gaussian process is characterized by the means $E[X(t)]$, & the covariances $\text{Cov}(X(s), X(t))$.

**Definition:** Standard Brownian motion is a Gaussian process with

$$E[X(t)] = 0 \quad \text{and} \quad \text{Cov}(X(s), X(t)) = \min\{s, t\}.$$ 

$$\Downarrow$$ $\text{Var}(X(t)) = E[X(t)^2] = t$

**Diffusion equations**

**Remark:** The probability density
Remark: The probability density
\[ f(x(t) = y \mid X(s) = x) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{2(t-s)}} \]
satisfies
\[ \frac{df}{dt} = \frac{1}{2} \frac{\partial^2 f}{\partial y^2} \]
forward diffusion equation
\[ \frac{df}{dt} = -\frac{1}{2} \frac{\partial^2 f}{\partial x^2} \]
backward diffusion equation
and is their unique solution.

These equations are easy to verify, or can be derived, following Einstein, by conditioning on \( X(t-h) \) or \( X(s+h) \), respectively, and letting \( h \to 0 \).

Example: Brownian bridge

Problem: For \( X(t) \) standard Brownian motion, what is the distribution of \( X(s) \) conditioned on \( X(t) = B \)?

Answer:
For \( s > t \), \( X(s) \) is a Gaussian process with mean \( B \),
\[ \text{Cov}(X(s), X(s_2)) = \min(s_1 - t, s_2 - t). \]
the same distribution as \( B + X(s-t) \).

For \( 0 \leq s \leq t \), \( X(s) \) is still a Gaussian process (starting with a multivariate Gaussian and conditioning on one of the variables, \( X(t) \), still leaves a Gaussian distribution).
The conditional density is
\[ f(x(s) = x \mid X(t) = B) = f_s(x) f_{t-s}(B-x) \]
\[ \propto \exp \left[ \frac{-x^2}{2s} - \frac{(B-x)^2}{2(t-s)} \right] \]
\[ \propto \exp \left[ \frac{-1}{2s(t-s)}(tx^2 - 2Bs) \right] \]
\[ \propto \exp \left[ -\frac{t}{2s(t-s)} (x - \frac{Bs}{t})^2 \right] \]
\[ \Rightarrow \mathbb{E}\left[ X(s) \mid X(t) = B \right] = \frac{Bs}{t} \]

\[
\text{Var}(X(s) \mid X(t) = B) = \frac{s(t-s)}{t} = t \cdot \frac{s}{t}(1 - \frac{s}{t})
\]

\[ \Rightarrow \text{Cov}(X(s_1), X(s_2) \mid X(t) = 0) \text{ for } s_1 < s_2 \]

\[ = \mathbb{E}\left[ X(s_1)(X(s_1) + (X(s_2) - X(s_1))) \mid X(t) = 0 \right] \]

\[ = \frac{s_1(t-s_1)}{t} + \mathbb{E}\left[ X(s_1)(X(s_2) - X(s_1)) \mid X(t) = 0 \right] \]

\[ = \frac{s_1(t-s_1)}{t} \left[ 1 - \frac{s_2-s_1}{t-s_1} \right] \]

\[ = \frac{s_1}{t}(t - s_2) \checkmark \]

**Observe:** Consider \( Y(t) = X(t) - t \cdot X(1) \).
For \( 0 \leq t \leq 1 \), \( Y(t) \) has the same dist as \( X(t) \mid X(1) = 0 \).

Indeed,
\[ \mathbb{E}[Y(t)] = 0 \]

and for \( 0 \leq s \leq t \leq 1 \),
\[ \text{Cov}(Y(s), Y(t)) = \mathbb{E}[\left( X(s) - s \cdot X(1) \right) \left( X(t) - t \cdot X(1) \right)] \]

\[ = \mathbb{E}\left[ X(s)X(t) - s \cdot X(t) \cdot X(1) - t \cdot X(s) \cdot X(1) + s \cdot t \cdot X(1)^2 \right] \]

\[ = s - s \cdot t - t \cdot s + s \cdot t \]

\[ = s(1 - s) \checkmark \]

**Exercise:** Verify that \( Z(t) = (1-t) \cdot X\left( \frac{t}{1-t} \right) \), \( Z(1) = 0 \), has the same dist as a Brownian bridge on \( [0,1] \).
Brownian motion with drift

\[ Y(t) = X(t) + \mu t \]

is a Gaussian process with \( \mathbb{E}[Y(t)] = \mu t \)

\( \text{Cov}(Y(s), Y(t)) = \min(s, t) \).
Hitting times for BM with drift

Just as for random walks, we can study the hitting times using martingales:

Claim: If \( X(t) \) is standard BM, then

\[
\begin{align*}
\cdot X(t) & \\
\cdot X(t)^2 - t & \\
\cdot e^{cX(t) - \frac{c^2}{2} t} & \\
\end{align*}
\]

are all martingales.

Proof for \( \exp[cX(t) - \frac{c^2}{2} t] \):

Recall: If \( X \sim N(\mu, \sigma^2) \), the mgf. is \( \mathbb{E}[e^{cX}] = e^{c\mu + \frac{c^2\sigma^2}{2}} \).

\[
\begin{align*}
\mathbb{E}[\exp(cX(t) - \frac{c^2}{2} t) | X(s)] \\
= e^{cX(s) - \frac{c^2}{2} t} \cdot \mathbb{E}[\exp(cX(t-s)) | X(s)] \\
= e^{cX(s) - \frac{c^2}{2} s} \cdot \mathbb{E}[\exp(cX(s)) | X(s)] \\
&= e^{cX(s) - \frac{c^2}{2} s} \cdot e^{rac{c^2}{2} s} \\
&= 1
\end{align*}
\]

\( Y(t) = X(t) + ut \)

\( T = \min\{t : Y(t) \in \mathcal{F} - A_1 + B_2\} \).

Martingale stopping

\[
\Rightarrow 0 = \mathbb{E}[X(0)] = \mathbb{E}[X(T)]
\]

\[
= p \mathbb{E}[B - \mu T | Y(T) = B] + (1-p) \mathbb{E}[-A - \mu T | Y(T) = -A]
\]

where

\[
\begin{align*}
p &= \mathbb{P}[Y(T) = B] \\
1-p &= \mathbb{P}[Y(T) = -A]
\end{align*}
\]

\[
= -\mu \mathbb{E}[T] - A + p(A + B)
\]
\[ E[T] = \frac{1}{\mu}(p(A+B)-A) \]

To find \( p \), use the third MG:

**MG stopping**

\[ 1 = E[\exp(cX(0) - \frac{c^2}{2}0)] \]
\[ = E[\exp(cX(T) - \frac{c^2}{2}T)] \]
\[ = c(Y(T) - \mu T) - \frac{c^2}{2}T \]
\[ = -2\mu Y(T) \text{ for } c = -2\mu \]
\[ = p \cdot e^{-2\mu B} + (1-p) e^{+2\mu A} \]

\[ \Rightarrow p = \frac{1-e^{+2\mu A}}{e^{-2\mu B} - e^{+2\mu A}} \]

*Observe:* If \( \mu < 0 \), letting \( A \to \infty \) we get

\( \Pr[Y(t) \text{ ever reaches } B] = e^{2\mu B} \)

If \( \mu > 0 \), \( \Pr[Y(t) \text{ ever reaches } B] = 1 \), and

\[ E[\text{time to reach } B] = \frac{1}{\mu}(p(A+B)-A) = \frac{B}{\mu} \]

**Geometric Brownian motion**

\[ Y(t) = e^{\sigma X(t)} \]

\[ E[Y(t)] = e^{\sigma^2 t/2}, \ Var(Y(t)) = E(Y(t)^2) - (EY(t))^2 = e^{2\sigma^2 t} - e^{\sigma^2 t} \]

**Example 1:** Value of a European call option:

Suppose a stock's price is given by

\[ S(t) = S_0 \cdot e^{\sigma X(t) + \mu t} \]
At time $T$ in the future, you have the option of buying the stock for price $K$.

What is the expected worth of the option?

$$
\mathbb{E}
\left[
\max
\left(\frac{1}{T}
\int_0^T S(t) \, dt - K, 0
\right)
\right]
= \int_{-\infty}^\infty \mathbb{E} \left[ e^{-x^2/2T} \cdot \max(0, S_0 e^{rt+\sigma T} - K) \right] \, dx
$$

for $x \geq \frac{1}{T} (\log \frac{K}{S_0} - \mu T)$

**Example 2:** Value of an Asian call option

$$
\max \left( 0, \frac{1}{T} \sum_{n=1}^T S(n) \right)
$$

To simulate this, use

$$
S(n+1) = S(n) \cdot e^{\mu + \sigma (X(n+1) - X(n))}.
$$

**Example 3:** A stock portfolio

What if you have two stocks?

$$
S_1(t) = S_1(0) \cdot e^{\sigma_1 X_1(t) + \mu_1 t}
$$

$$
S_2(t) = S_2(0) \cdot e^{\sigma_2 X_2(t) + \mu_2 t}
$$

$X_1(t)$ & $X_2(t)$ can be independent std. BM.

But what if changes in stock prices are correlated?

**Observe:** If $X, Y \sim N(0,1)$, $X \perp Y$, then $Z = \cos \Theta X + \sin \Theta Y \sim N(0,1)$

with $\text{cov}(X, Z) = \cos \Theta$.

$\Rightarrow$ If $X(t) \perp Y(t)$ are standard BM processes,

$Z(t) = \cos \Theta X(t) + \sin \Theta Y(t)$ is std BM,

with $\text{cov} \left( X(s), Z(t) \right) = \cos \Theta \cdot \text{cov}(X(s), X(t)) = \cos \Theta \cdot \min(s,t)$. 


Example: If \( \mu_1 = .01 \), \( \sigma_1^2 = 1 \), \( \mu_2 = .02 \), \( \sigma_2^2 = 2 \),
\[
\text{Cov}(X_1(t), X_2(t)) = \sqrt{2} \cdot t,
\]
\( S_1(0) = 1 \), \( S_2(0) = 2 \),
what is
\[
\mathbb{P} \left[ S_1(10) + S_2(10) > 1.1 (S_1(0) + S_2(0)) \right]?
\]

Answer: While we can get a closed form involving a double integral, it is more practical just to simulate it:

\[
t = 10;
\mu_1 = .01; \sigma_1 = 1;
\mu_2 = .02; \sigma_2 = \sqrt{2};
S_1 = 1;
S_2 = 2;
\]
\[
\text{numtrials} = 10^6;
wins = 0;
\text{For} [\text{trial} = 1, \text{trial} \leq \text{numtrials}, \text{trial}++,
\{
\{x, y\} = \text{RandomVariate}[\text{NormalDistribution}[0, \sqrt{t}], 2];
\]
\[
z = \frac{1}{\sqrt{2}} (x + y);
\]
\[
\text{If} [S_1 e^{\sigma_1 \cdot x + \mu_1 \cdot t} + S_2 e^{\sigma_2 \cdot x + \mu_2 \cdot t} > 1.1 (S_1 + S_2), \text{wins}++];
\]
\]
\[
\text{wins} / \text{numtrials} \quad // \quad \text{N}
\]
\[
0.570815
\]

Remark: To simulate standard BM at times \( 0 = t_0 < t_1 < t_2 < \cdots < t_k \),
generate \( Z_1, Z_2, \ldots, Z_k \sim \text{N}(0, 1) \) iid.
and let
\[
X(t_j) = \sum_{i=1}^{j} \sqrt{t_i - t_{i-1}} Z_i
= X(t_{j-1}) + \sqrt{t_i - t_{i-1}} Z_i \quad \checkmark
\]

Example: Value of a perpetual American call option
Suppose the price of a stock is given by
\[ S(t) = S_0 \cdot \exp(\sigma \cdot X(t) - \mu t) \]

where \( X \) is standard BM and \( \mu > 0 \).

We are given the option of buying the stock at price \( P \), at any time in the future.

When should we exercise the option?, and

What is our expected return?

Answer:

The profit from using the option is \( S(t) - P \).

Obviously, we shouldn't use the option if \( S(t) < P \).

But when should we use it?

Consider the policy: use the option if \( S(t) = Q \).

\[
E[\text{profit}] = (Q - P) \cdot \mathbb{P}[S(t) \text{ ever reaches } Q]
\]

\[
= \exp \left( \frac{\mu t}{\sigma} \cdot \log Q \right)
\]

\[
= \exp \left( \frac{2\mu}{\sigma^2} \cdot \log Q \right) = Q^{-2\mu/\sigma^2}
\]

\[
E[\text{profit}] = (Q - P) \cdot Q^{-2\mu/\sigma^2}
\]

Now maximize over \( Q \):

\[
D \left[ (Q - P) \cdot Q^{-2\mu/\sigma^2}, Q \right] \quad \text{// FullSimplify}
\]

\[
\text{Solve}[\% = 0, Q]
\]

\[
Q^{-\frac{2\mu}{\sigma^2}} \left( 2P \mu + Q \left( -2 \mu + \sigma^2 \right) \right) = 0
\]

\[
\left\{ Q \rightarrow \frac{2P \mu}{\sigma^2} \right\} = P \cdot \frac{1}{1 - \frac{\sigma^2}{2\mu}}
\]
observe \( G \) increases with volatility \( \sigma \)
decreases with drift \( \mu \).

**Brownian motion reflected at the origin**

\[
Z(t) = |X(t)|
\]

\( t \) standard BM

\[
f_{Z(t)}(z) = 2f_{X(t)}(z)
\quad \text{for } z \geq 0
\]

\[
= \frac{2}{\sqrt{2\pi t}} e^{-z^2/(2t)}
\]

\[
\Rightarrow \mathbb{E}[Z(t)] = \int_0^\infty z f_{Z(t)}(z) \, dz = \sqrt{\frac{2t}{\pi}}
\]

\[
\text{Var}(Z(t)) = (1-\frac{2}{\pi})t.
\]

**Claim:** \(|X(t)|\) and \( \max_{0 \leq s \leq t} X(s) \) have the same distribution.

**Proof:**

\[
P\left( \max_{0 \leq s \leq t} X(s) \geq z \right) = P\left[ T_z \leq t \right]
\quad \text{(by continuity)}
\]

\[
= 2P\left[ X(t) \geq z \right]
\]

\[
= 2 \frac{1}{\sqrt{2\pi t}} \int_z^\infty e^{-x^2/(2t)} \, dx
\]

\[
P\left( \max_{0 \leq s \leq t} X(s) \leq z \right) = 2 \frac{1}{\sqrt{2\pi t}} \int_0^z e^{-x^2/(2t)} \, dx
\]

**Zeros of Brownian motion, and arc-sin laws** \[\text{[Ross §8.2]}\]

**Theorem:** The probability that a standard BM process \( X \)
has a zero in the time interval \( (t_0, t_1) \) is

\[
\frac{2}{\pi} \cos^{-1}\left(\sqrt{\frac{t_0}{t_1}}\right)
\]

(The probability there's no zero is \( \frac{2}{\pi} \sin^{-1}\left(\sqrt{\frac{t_0}{t_1}}\right) \).)
Consequences:

- $P\left[\text{there exists a } 0 \text{ in } (0, t)\right] = 1$ for all $t > 0$
- $\inf \{ t > 0 : X(t) = 0 \} = 0$ almost surely.
- There are infinitely many zeros in $[0, t]$ almost surely.

Proof of the theorem:

Let $E$ be the event that there's a zero in $(t_0, t)$.

Conditioning on $X(t_0)$

$$P[E] = \int_{-\infty}^{\infty} P[E|X(t_0) = x] dx$$

$$= \int_{-\infty}^{\infty} P[T_x \leq t_1 - t_0]$$

$$= P[T_x \leq t_1 - t_0]$$

$$= 2P[X(t_1 - t_0) > |x|]$$

$$= \frac{2}{\sqrt{\pi t_0}} \int_0^\infty \frac{x^2}{(2(\pi t_0)^{3/2})} \int_0^\infty dz e^{-\frac{z^2}{2(t_1 - t_0)}}$$
\[
\frac{2}{\sqrt{2\pi t_0}} \int_0^\infty \left( \frac{2}{\sqrt{2\pi (t_1 - t_0)}} \right) \int_x^\infty e^{-\frac{z^2}{2(t_1 - t_0)}} \, dz \, dx \quad \text{// Simplify[#, \{0 < t_0 < t_1\}] &
\]

\[
2 \tan^{-1} \left( \frac{\xi}{\nu} \right)
\]

and \( \tan^{-1} \sqrt{-1 + \frac{1}{x}} = \cos^{-1} \sqrt{x} \), since

\[
\tan \cos^{-1} \sqrt{x} = \frac{\sin(\cos^{-1} \sqrt{x})}{\cos(\cos^{-1} \sqrt{x})} = \sqrt{\frac{1-x}{x}} = \sqrt{-1 + \frac{1}{x}}
\]

\qed