Toward a meta-stable range
in \(A^1\)-homotopy theory of punctured affine spaces
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Suppose \(k\) is a perfect field having characteristic unequal to 2. Write \(S_k\) for the category of schemes that are separated, smooth and of finite type over \(k\). Write \(H^\cdot(k)\) for the Morel-Voevodsky unstable pointed \(A^1\)-homotopy category \([MoVo99]\). A (pointed) \(k\)-space \(X\) (resp. \((X,x)\)) is a (pointed) simplicial Nisnevich sheaf on \(S_k\). Given two pointed \(k\)-spaces \((X,x)\) and \((Y,y)\), we write \([X,Y]\) for \(\text{Hom}_{H^\cdot(k)}(X,Y)\). If \((X,x)\) is a pointed \(k\)-space, write \(\pi_{A^1}^\cdot(X,x)\) for the Nisnevich sheaf associated with the presheaf \(U \mapsto S_k^\wedge U_+ \rightarrow (X,x)\).

Point \(A^n\) by \((1,0,\ldots,0)\), and suppress this base-point from notation. Results of Morel yield a description of \(\pi_{A^1}^{n-1}(A^n \setminus 0), n \geq 2,\) as the sheaf \(K_n^{MW}\) of “unramified Milnor-Witt K-theory.” In previous work, the authors provided a description of \(\pi_{A^1}^1(A^2 \setminus 0)\) and \(\pi_{A^1}^2(A^3 \setminus 0)\) \([AsFa12a, AsFa12b]\). The goal of the talk was to provide a conjectural description of \(\pi_{A^1}^n(A^n \setminus 0)\) for \(n \geq 4\). The proposed description is in two parts.

### Suslin matrices and the degree map

Schlichting and Tripathi constructed an orthogonal Grassmannian \(OGr\) and showed that \(Z \times OGr\) represents Hermitian K-theory in the unstable \(A^1\)-homotopy category \([ScTr12]\). They also establish a geometric form of Bott periodicity in Hermitian K-theory that identifies various loop spaces of \(Z \times OGr\) in terms of other natural spaces; we summarize this result as follows.

**Proposition 1.** There are weak equivalences of the form

\[
\Omega^1_{\ast}O_{Z \times OGr} \xrightarrow{\sim} \begin{cases} 
O & \text{if } i \equiv 0 \pmod{4} \\
GL/Sp & \text{if } i \equiv 1 \pmod{4} \\
Sp & \text{if } i \equiv 2 \pmod{4}, \text{ and} \\
GL/O & \text{if } i \equiv 3 \pmod{4}; 
\end{cases}
\]

Here \(O := \text{colim}_n O(q_{2n})\), where \(q_{2n}\) is the standard hyperbolic form, \(Sp := \text{colim}_n Sp_{2n}\), \(GL/Sp := \text{colim}_n GL_{2n}/Sp_{2n}\), and \(GL/O := \text{colim}_n GL_{2n}/O(q_{2n})\).

The class of \(\langle 1 \rangle \in GW(k)\) yields a distinguished element in \(GW(k) = [\text{Spec } k_+, Z \times OGr]_{A^1}\). An adjunction argument can be used to show that this element corresponds to a distinguished class in \([A^n \setminus 0, P_n]_{A^1}\), where \(P_n\) is either \(O, GL/O, Sp\) or \(GL/Sp\) depending on whether \(n\) is congruent to 0, 1, 2 or 3 modulo 4.
Let $Q_{2n-1}$ be the smooth affine quadric defined as a hypersurface in $\mathbb{A}^{2n}$ given by the equation $\sum_{i} x_{i}z_{n+i} = 1$. There is an $\mathbb{A}^1$-weak equivalence $Q_{2n-1} \to \mathbb{A}^n \setminus 0$ defined by projecting onto the first $n$ variables. Each variety $P_n$ is an ind-algebraic variety, and Suslin inductively defined certain matrices $S_n$ that correspond to morphisms $s_n : Q_{2n-1} \to P_n$.

**Proposition 2.** The distinguished homotopy classes $[\mathbb{A}^n \setminus 0, P_n]_{\mathbb{A}^1}$ described in the previous paragraph is represented by the morphism $s_n : Q_{2n-1} \to P_n$ given by the matrix $S_n$.

The $\mathbb{A}^1$-homotopy sheaves of $O, GL/O, Sp$ and $GL/Sp$ can be identified in terms of the Nisnevich sheafification of Schlichting’s higher Grothendieck-Witt groups. Indeed, $\pi_{i}^{A^1}(O) \cong GW_{i+1}^0$, $\pi_{i}^{A^1}(GL/O) \cong GW_{i+1}^1$, $\pi_{i}^{A^1}(Sp) \cong GW_{i+1}^2$ and $\pi_{i}^{A^1}(GL/Sp) \cong GW_{i+1}^3$. In general, the sheaves $GW_j^i$ are viewed as 4 periodic in $j$. Therefore, the morphism $s_n$ yields, upon applying the functor $\pi_{n}^{A^1}(\cdot)$, a morphism

$$s_{n*} : \pi_{n}^{A^1}(\mathbb{A}^n \setminus 0) \longrightarrow GW^n_{n+1}.$$ 

This morphism is not surjective for $n \geq 4$, but it does coincide with a corresponding morphism constructed in the computations of $\pi_{2}^{A^1}(\mathbb{A}^2 \setminus 0)$ and $\pi_{3}^{A^1}(\mathbb{A}^3 \setminus 0)$.

Recall the contraction of a sheaf $\mathcal{F}$ is defined by the formula $\mathcal{F}_{-1}(U) := \ker((id \times e)^* : \mathcal{F}(G_m \times U) \to \mathcal{F}(U))$, where $e : Spec \, k \to G_m$ is the unit section. One defines $\mathcal{F}_{-i}$ inductively as $(\mathcal{F}_{-(i-1)})_{-1}$.

**Theorem 3.** The morphism $s_{n*}$ becomes surjective after $(n-3)$-fold contraction and split surjective after $n$-fold contraction.

### Motivic Hopf maps and the kernel of the degree map

In [AsFa12], we introduced the geometric Hopf map $\nu : \mathbb{A}^4 \setminus 0 \to \mathbb{P}^1\wedge 2$ and showed that it was $\mathbb{P}^1$-stably essential (i.e., is not null $\mathbb{A}^1$-homotopic after repeated $\mathbb{P}^1$-suspension). For any integer $n \geq 2$, set

$$\nu_n := \sum_{d|n} \nu : \mathbb{A}^{n+2} \setminus 0 \longrightarrow \mathbb{P}^1\wedge n.$$ 

Applying $\pi_{n}^{A^1}(\cdot)$ to the above morphism yields a map

$$(\nu_n)_* : K_{n+2}^{MW} \longrightarrow \pi_{n+1}^{A^1}(\mathbb{P}^1\wedge n).$$

For $n \geq 4$, Morel’s Freudenthal suspension theorem yields isomorphisms

$$\pi_{n}^{A^1}(\mathbb{A}^n \setminus 0) \sim \pi_{n+1}^{A^1}(\mathbb{P}^1\wedge n),$$

so in this range, we can view $(\nu_n)_*$ as giving a map $K_{n+2}^{MW} \to \pi_{n}^{A^1}(\mathbb{A}^n \setminus 0)$.

For $n = 3$, Morel’s Freudenthal suspension theorem only yields an epimorphism. We can refine this result to provide an analog of the beginning of the EHP sequence in $\mathbb{A}^1$-homotopy theory. A particular case of the general result we can establish can be stated as follows.
Theorem 4. There is an exact sequence of the form
\[ \pi_{A^1}(\mathbb{P}^{1^\wedge 3}) \xrightarrow{H} \pi_{A^1}(\Sigma^1_3(\mathbb{A}^{3^\wedge 0})) \xrightarrow{P} \pi_{A^1}(\mathbb{A}^{3^\wedge 0}) \xrightarrow{E} \pi_{A^1}(\mathbb{P}^{1^\wedge 3}) \rightarrow 0. \]

The morphism $H$ in the above exact sequence conjecturally admits a description as a variant of the Hopf invariant in Chow-Witt theory. Assuming this, the results we have proven on $\pi_{A^1}(\mathbb{A}^{3^\wedge 0})$ show that $\nu_{3*}$ factors through an explicit quotient of $K^{MW}_5$. In turn, this (conjectural) computation suggests the following conjecture.

Conjecture 5. For any integer $n \geq 3$, the morphism $\nu_{n*}$ factors through a morphism $K^{MW}_{n+2}/24 \rightarrow \pi_{A^1}^{n}(\mathbb{P}^{1^\wedge n})$.

The structure of $\pi_{A^1}^{n}(\mathbb{A}^{n^\wedge 0})$

We now study the relationship between the two morphisms constructed above. Using an obstruction theory argument, one can demonstrate the following result.

Proposition 6. For any integer $n \geq 4$, the composite map
\[ K^{MW}_{n+2} \rightarrow \pi_{A^1}^{n}(\mathbb{A}^{n^\wedge 0}) \rightarrow GW^{n}_{n+1} \]
is zero.

Combining everything discussed so far, one is led to the following conjecture.

Conjecture 7. For any integer $n \geq 4$, there is an exact sequence of sheaves of the form
\[ K^{MW}_{n+2}/24 \rightarrow \pi_{A^1}^{n}(\mathbb{A}^{n^\wedge 0}) \rightarrow GW^{n}_{n+1}. \]
The sequence becomes short exact after $n$-fold contraction.

Remark 8. The conjecture above stabilizes to an unpublished conjecture of F. Morel on the stable motivic $\pi_1$ of the motivic sphere spectrum. Using the motivic Adams(-Novikov) spectral sequence, K. Ormsby and P.-A. Østvær have checked that after taking sections over fields having 2-cohomological dimension $\leq 2$, the 2-primary part of the stable conjecture is true. Nevertheless, the stable conjecture does not imply the conjecture above (even for large $n$) because of a lack of a Freudenthal suspension theorem for $P^1$-suspension. On the other hand, the conjecture above for every $n$ sufficiently large implies the stable conjecture.

Remark 9. By the results of [AsFa12b], the above conjecture immediately implies “Murthy’s conjecture:” if $X$ is a smooth affine $(d + 1)$-fold over an algebraically closed field $k$, and $E$ is a rank $d$ vector bundle on $X$, then $E$ splits off a free rank 1 summand if and only if $0 = c_d(E) \in CH^{d}(X)$. However, the conjecture is much stronger: it gives the complete secondary obstruction to splitting a free rank 1 summand of a vector bundle on a smooth affine scheme.
References


