Rational points vs. 0-cycles of degree 1 in stable $\mathbb{A}^1$-homotopy

Aravind Asok, Christian H"asemeyer and Fabien Morel

Suppose $k$ is a field, and $X$ is a smooth variety over $k$. Let $\mathcal{H}(k)$ denote the $\mathbb{A}^1$-homotopy category of smooth schemes over $k$ \cite{MV99}; abusing notation, we write $X$ for the isomorphism class of a smooth scheme in $\mathcal{H}(k)$. Let $\mathcal{SH}(k)$ denote the stable $\mathbb{A}^1$-homotopy category of smooth schemes over $k$, i.e., the category of $\mathbb{P}^1$-spectra over $k$ \cite{Mor05}. The suspension spectrum $\Sigma_\mathbb{P}^\infty \text{Spec } k_+$, denoted $S^0$ for notational convenience, is called the motivic sphere spectrum.

If $U$ is another smooth variety, write $[U, X]_{\mathbb{A}^1}$ for the set $\text{Hom}_{\mathcal{H}(k)}(U, X)$ and write $[U, X]_{\text{st}}$ for the abelian group $\text{Hom}_{\mathcal{SH}(k)}(\Sigma_\mathbb{P}^\infty U_+, \Sigma_\mathbb{P}^\infty X_+)$. Define $\pi^1_0(X)$ to be the Nisnevich sheaf on $\text{Sm}_k$ associated with the presheaf $U \mapsto [U, X]_{\mathbb{A}^1}$ and $\pi^0_0(X)$ to be the Nisnevich sheaf on $\text{Sm}_k$ associated with the presheaf $U \mapsto [U, X]_{\text{st}}$. Each of these sheaves determines “by restriction” a functor on the category of finitely generated separable extensions $L/k$.

Stable homotopy theory and rational points

If $\pi^1_0(X)(k)$ is non-empty, we say that $X$ has a rational point up to unstable $\mathbb{A}^1$-homotopy. It is known that if $X$ has a rational point up to unstable $\mathbb{A}^1$-homotopy, then $X$ has a rational point \cite{MV99}. Thus, existence of a rational point is an unstable $\mathbb{A}^1$-homotopy invariant.

Similarly, say that $X$ has a rational point up to stable $\mathbb{A}^1$-homotopy if the canonical map $\pi^0_0(X) \to \pi^0_0(S^0)$ is a split epimorphism; a choice of a splitting will be called a rational point up to stable $\mathbb{A}^1$-homotopy. Any rational point up to unstable $\mathbb{A}^1$-homotopy determines a rational point up to stable $\mathbb{A}^1$-homotopy by taking iterated $\mathbb{P}^1$-suspensions. If $X$ is smooth and proper, there is a group homomorphism from $\pi^0_0(X)(k)$ to the group of 0-cycles of degree 1; a priori it is not clear that this map is either surjective or injective.

Theorem 1. Assume $k$ is a field having characteristic 0. If $X$ is a smooth proper $k$-variety, then $X$ has a 0-cycle of degree 1 if and only if $X$ has a rational point up to stable $\mathbb{A}^1$-homotopy.

Sheaves of connected components

We deduce the above result from a description of the sheaf $\pi^0_0(X)$ for any smooth proper variety. The description is motivated by foundational work of Morel describing the sheaf $\pi^0_0(S^0)$ in terms of the Grothendieck-Witt ring \cite{Mor04}. There is a “Hurewicz” functor from the stable $\mathbb{A}^1$-homotopy category to Voevodsky’s derived category of motives. The analog of the stable $\pi^0$ computed in Voevodsky’s derived category of motives is the 0-th Suslin homology sheaf. For a smooth proper variety $X$, the sections of this sheaf over fields coincide with the Chow group of 0-cycles on $X_L$ (cf. \cite{Deg08 §3.4}).
We use the theory of oriented Chow groups, or Chow-Witt groups, as invented by J. Barge and F. Morel [BM00], and developed in detail by J. Fasel [Fas08, Fas07]. For any \(n\)-dimensional smooth proper \(k\)-scheme \(X\), one can define the oriented Chow group \(\tilde{\text{CH}}_0(X)\) by means of a certain “oriented Chow cohomology group” \(\tilde{CH}^n(X, \omega_X)\) (see [Fas08, Definition 10.2.17] for details). This latter group is defined by means of an explicit Gersten resolution, and has functorial pushforwards for proper morphisms.

**Theorem 2.** If \(X\) is a smooth proper \(k\)-variety over a field \(k\) having characteristic 0, then there is an isomorphism (natural with respect to \(X\)) between the functor \(L \mapsto \pi^s_0(X)(L)\) and the functor \(L \mapsto \tilde{CH}_0(X_L)\).

**Sketch of proof of Theorem 2.** One first reduces to the case where \(X\) is projective, and deals with an associated “abelianized” problem using a version of \(\mathbb{A}^1\)-homology that has been stabilized with respect to \(\mathbb{G}_m\). When \(X\) is projective, the idea of the proof is to use Spanier-Whitehead duality: the Spanier-Whitehead dual of a smooth scheme \(X\) is the Thom space of the negative tangent bundle (see, e.g., [Hu05, Theorem A.1] or [Rio05, Théorème 2.2]).

When \(X\) has trivial tangent bundle, one can prove the result by proving a \(\mathbb{P}^1\)-bundle formula for the oriented Chow group of 0-cycles—this involves some facts about contractions of the sheaf \(K_M^{MW}\) as discussed at the end of [Mor05, §2.3]. In the general case, one has to show that the twist arising from non-triviality of the negative tangent bundle only appears through the canonical bundle \(\omega_X\) of \(X\). Locally the tangent bundle is trivial, and a careful patching argument (using the fact that any element of \(GL_n\) is \(\mathbb{A}^1\)-homotopic to its determinant) can be used to finish the proof; this involves an “unstable” construction of the map inducing duality as given by Voevodsky in [Voe03].

**Sketch of proof of Theorem 3.** The “only if” direction is straightforward. For the “if” direction, it suffices to show that the “forgetful” morphism \(\tilde{CH}_0(X_L) \rightarrow CH_0(X_L)\)—functorial in \(L\) and \(X\)—is always a surjection. For any field \(F\), the canonical map \(GW(F) \rightarrow \mathbb{Z}\) given by the rank homomorphism is always surjective. Each of these groups is computed by means of a Gersten resolution. One then just uses the fact that \(X\) has Nisnevich cohomological dimension \(n\).

**Remark 3.** In fact, we prove a more precise result. The sheaf \(\pi^s_0(X)\) is a strictly \(\mathbb{A}^1\)-invariant sheaf of groups by [Mor05, Theorem 6.2.7] and therefore “unramified” in an appropriate sense; one can then describe the sections of the sheaf \(\pi^s_0(X)\) over a smooth scheme \(U\) in terms of sections over \(k(U)\) together with information coming from discrete valuations associated with codimension 1 points of \(U\).

**References**


