Rational connectivity and $\mathbb{A}^1$-connectivity
or geometric applications of the Milnor conjectures
(joint with F. Morel)

Aravind Asok (UCLA)

May 8, 2009
The goal

“No doubt topologists will welcome a version which can be read by those not familiar with modern algebraic geometry.”

-J.F. Adams
from Math Reviews
1 Conventions, definitions and basic examples
Outline

1. Conventions, definitions and basic examples
2. An elementary example
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2 An elementary example

3 A proposed generalization
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2 An elementary example
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4 The geometric/topological mechanism
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  - take $L = \mathbb{C}(t_1, \ldots, t_n)$ and think of a family of varieties.
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All algebraic varieties will be assumed smooth, connected, and often proper (read: compact).
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All algebraic varieties will be assumed smooth, connected, and often proper (read: compact).

Given an algebraic variety $X$ over $L$, we write $L(X)$ for its field of rational functions.
Basic definitions: rationality

**Definition**

An algebraic variety $X$ over $L$ is *$L$-rational* if $L(X) \cong L(t_1, \ldots, t_n)$. 

Write $\mathbb{P}^n$ for $n$-dimensional projective space (over $L$), which is the basic example of a rational variety. Think: “most,” i.e., a (Zariski) open set, of the solutions to the equations defining $X$ can be rationally parameterized.
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Basic question

**Question**

*If* \( X_d \subset \mathbb{P}^n_{\mathbb{C}} \) *is a smooth degree* \( d \) *complex hypersurface, (when) is* \( X_d \) *rational?*
Basic example

Example

If $X_2 \subset \mathbb{P}^n_C$, i.e., a quadric, then $X_2$ is rational.
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- Same argument shows any quadric over a field $F$ having an $F$-rational point is actually $F$-rational.
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What about the case $d = 3, \ n = 2$?

- This argument fails for smooth cubic curves in $\mathbb{P}_\mathbb{C}^2$.
- Of course, there are many ways to prove that smooth cubic curves are not rational, but let us give another (elementary) argument.
What about the case $d = 3$, $n = 2$?

- This argument fails for smooth cubic curves in $\mathbb{P}^2_C$.
- Of course, there are many ways to prove that smooth cubic curves are not rational, but let us give another (elementary) argument.
- Begin by defining another invariant.
Fields and valuations

Let $L/\mathbb{C}$ be a finitely generated extension, and let $L^\ast$ denote the multiplicative group of non-zero elements. A discrete valuation is a group homomorphism $\nu: L^\ast \to \mathbb{Z}$ satisfying a "metric" property. Write $V(L)$ for the set of inequivalent discrete valuations of $L$. Any discrete valuation $\nu$ gives rise to a homomorphism $\partial \nu: L^\ast / (L^\ast)^2 \to \mathbb{Z} / 2\mathbb{Z}$. 

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$$\partial_\nu : L^*/(L^*)^2 \to \mathbb{Z}/2\mathbb{Z}.$$
Unramified square classes

Definition

\[ \kappa_{ur}^1(\mathbb{L}/\mathbb{C}) := \bigcap_{\nu \in V(\mathbb{L})} \ker(\partial_{\nu} : \mathbb{L}^* / (\mathbb{L}^*)^2 \to \mathbb{Z}/2) \].

Elements of \( \kappa_{ur}^1(\mathbb{L}/\mathbb{C}) \) will be referred to as unramified (square) classes, or simply unramified elements, and \( \kappa_{ur}^1(\mathbb{L}/\mathbb{C}) \) will be called the group of unramified square classes.
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\[ k_1^{ur}(L/\mathbb{C}) := \bigcap_{\nu \in V(L)} \ker(\partial_\nu : L^*/(L^*)^2 \to \mathbb{Z}/2). \]
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  - Why? Every class in $\mathbb{C}(t)^*/(\mathbb{C}(t)^*)^2$ admits a representative lying in $\mathbb{C}[t]$; use the fundamental theorem of algebra.
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- If $L = \mathbb{C}(t)$, then $k_1^{ur}(L/\mathbb{C}) = 0$.
  - Why? Every class in $\mathbb{C}(t)^*/(\mathbb{C}(t)^*)^2$ admits a representative lying in $\mathbb{C}[t]$; use the fundamental theorem of algebra.
- In fact, $k_1^{ur}(\mathbb{C}(t_1, \ldots, t_n)/\mathbb{C}) = 0$. 
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$$\mathbb{C}(x)^* / (\mathbb{C}(x)^*)^2 \longrightarrow L^* / (L^*)^2$$

whose kernel is generated by $f$. 

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Rational connectivity and $\mathbb{A}^1$-connectivity
An exact sequence

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- The field extension $\mathbb{C}(x) \hookrightarrow L$ gives rise to an exact sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{C}(x)^*/(\mathbb{C}(x)^*)^2 \longrightarrow L^*/(L^*)^2$$

sending $1 \in \mathbb{Z}/2\mathbb{Z}$ to the image of $f$ in $\mathbb{C}(x)^*/(\mathbb{C}(x)^*)^2$. 

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  - \( f_x = (x + 1)(x - 1) \) is a square.
An example (continued)

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Exc: Using the equation $y^2 = x(x+1)(x-1)$, show that $2\nu(y) = \nu(x)$. 

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*Note:* with more work, one can actually determine the group $k_1^{ur}(L/\mathbb{C})$. 

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- **Case** $d = 3$, $n > 4$. No known irrational examples, though some rational examples *are* known (Hassett ’99)!
A reformulation

- Assume $n > 3$. 

**Remark** If the cubic hypersurface is “more special,” i.e., it possesses a linear subspace of higher dimension, then one can equip it with the structure of a higher dimensional quadric bundle.
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Remark

If the cubic hypersurface is “more special,” i.e., it possesses a linear subspace of higher dimension, then one can equip it with the structure of a higher dimensional quadric bundle.
First observations

We’ll look at the rationality problem for quadric bundles as above, which we can also think of as quadrics over $\mathbb{C}(t_1, \ldots, t_n)$. 

All “elementary” birational invariants of these higher dimensional quadric bundles are trivial. These varieties are rationally connected in the sense of Campana-Kollár-Miyaoka-Mori, i.e., any two $\mathbb{C}$-points can be connected by a $\mathbb{P}^1$. This implies their topological fundamental group is trivial, and, e.g., these varieties have no non-zero holomorphic $m$-forms. The group $\text{kur}^1(L/\mathbb{C})$ is trivial for any of these varieties. What prevents these varieties from being rational? The quadric bundle need not admit a (rational) section!
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What prevents these varieties from being rational? The quadric bundle need not admit a (rational) section!
We will define "higher" versions of $k_{ur}$ that have a better chance of being non-trivial.

Observe:

$L^*: = K^1(L)$, and $L^*/(L^*)^2 = K^1(L)/2$.

One possible generalization of the group of square classes goes by way of higher Milnor K-theory.

The maps induced by discrete valuations can be thought of as "residue" maps in Milnor K-theory.
Higher unramified invariants

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Given a field $L$, set

$$K_*(L) := T_\mathbb{Z}(L^*)/J,$$

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Milnor K-theory
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Aravind Asok (UCLA) 
Rational connectivity and $\mathbb{A}^1$-connectivity
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- Let $K_n^M(L)$ denote the $n$-th graded piece of this ring.
Milnor K-theory

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Given a field $L$, set

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where $T_\mathbb{Z}(L^*)$ denotes the tensor algebra on $L^*$, and $J$ denotes the Steinberg ideal, i.e., the graded ideal generated by $a \otimes (1 - a)$ for $a \neq 0, 1$.

- Let $K^M_n(L)$ denote the $n$-th graded piece of this ring.
- Set $k_n(L) := \text{coker}(K^M_n(L) \xrightarrow{\times 2} K^M_n(L))$; we call this mod 2 Milnor K-theory.
Residue maps

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Residue maps

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Aravind Asok (UCLA)  
Rational connectivity and $\mathbb{A}^1$-connectivity
Unramified mod 2 Milnor K-theory

Definition

\[ k_{\text{ur}}(L/C) := \bigcap_{\nu \in V}(L) (\ker(\partial_\nu : k_n(L) \to k_n-1(\kappa_\nu)), \text{and call this group the unramified mod 2 Milnor K-theory of } L. \]

One can check \( k_{\text{ur}}(L/C) \) is an invariant of \( L/C \), \( k_{\text{ur}}(L/C) \) is a covariant functor on field extensions, and \( k_{\text{ur}}(\mathbb{C}(t_1, \ldots, t_n)/\mathbb{C}) = 0 \).

Goal: apply this invariant to study rationality of quadric bundles.
Unramified mod 2 Milnor K-theory

Definition

Set

\[ k^ur_n(L/\mathbb{C}) := \bigcap_{\nu \in \mathcal{V}(L)} (\ker(\partial_\nu : k_n(L) \longrightarrow k_{n-1}(\kappa_\nu)), \]

where \( \mathcal{V}(L) \) is the set of valuations of \( L \).
Unramified mod 2 Milnor K-theory

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Goal: apply this invariant to study rationality of quadric bundles.
An exact sequence

Recall that if $L = \mathbb{C}(x)(\sqrt{f})$, with $y^2 = f(x)$ we had

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow k_1(\mathbb{C}(x)) \longrightarrow k_1(L),$$

where the kernel is generated by $f$. 
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- If $F$ is a field, and $f, g \in F^*$, consider the conic $x^2 + fy^2 = gz^2$; denote it $Q(f,g)$.
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If $F$ is a field, and $f, g \in F^*$, consider the conic $x^2 + fy^2 = gz^2$; denote it $Q_{(f,g)}$. Functoriality gives a map:

$$k_i(F) \rightarrow k_i(F(Q_{(f,g)})).$$
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**Question**: Can one describe the kernel of this map?
An exact sequence (continued)

The pair \((f, g)\) determines an element of \(k_2(F)\), which we refer to as the symbol \((f, g)\).
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**Theorem (Amitsur ’55 + (many authors) + Merkurjev ’81)**

The kernel of

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- Use this to study rationality problems.
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- Generalize this result.
The Milnor conjecture
The Milnor conjecture

“So you’re telling me that two groups, both of which are really hard to understand, are isomorphic?”

- Anonymous
Some more notation

Notation:

- Take $a_1, \ldots, a_n \in F^\ast$.
- Write $\langle a_1, \ldots, a_n \rangle$ for the quadratic form $a_1 x_1^2 + \cdots + a_n x_n^2$.
- Set $\langle\langle a_1, \ldots, a_n \rangle\rangle := \langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$.
- Write $Q(a_1, \ldots, a_n)$ for the (small Pfister) quadric defined by the equation $\langle\langle a_1, \ldots, a_n - 1 \rangle\rangle = \langle a_n \rangle$.

Example: When $n = 1$, such quadrics are given by the equation $y^2 = f$. When $n = 2$, such quadrics reduce to the conics from before.
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Some quick (revisionist) history

**Goal**: study the kernel of the map $k_n(F) \to k_n(F(Q(a_1,\ldots,a_n)))$. 
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**Note:** $(a_1,\ldots,a_n)$ determines an element of $k_n(F)$, which we call the associated symbol; easy to show that $(a_1,\ldots,a_n)$ is contained in the kernel.
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- \( n = 4 \), (Jacob-Rost ’89 + ⋅⋅⋅) proved that the kernel is generated by the symbol.
A consequence of the Milnor conjecture

The kernel of the map $k_n(F) \to k_n(F(Q(a_1,...,a_n)))$ is generated by $(a_1,...,a_n)$.

Some key points in the proof.

"Topological" part: Voevodsky's construction and study of properties of Steenrod operations on an appropriately defined cohomology theory for algebraic varieties.

"Geometric" part: Rost's study of small Pfister quadrics.
Conventions, definitions and basic examples
An elementary example
A proposed generalization
The geometric/topological mechanism

The problem revisited
Generalizing Step 1: defining higher invariants
Generalizing Step 2: constructing an exact sequence
Generalizing Step 3: constructing unramified elements

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**Theorem (Orlov-Vishik-Voevodsky ’07)**

*The kernel of the map* $k_n(F) \to k_n(F(Q(a_1,\ldots,a_n)))$ *is generated by the symbol* $(a_1,\ldots,a_n)$. 
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Application to rationality problems I

Example (Non-rational conic bundles)

Artin-Mumford '71, Colliot-Thélène-Ojanguren '89; Take $L = \mathbb{C}(x_1, x_2)$. Take $f, g_1, g_2$ in $L^*$, and consider the conic $Q(f, g_1 g_2)$. For appropriate choice of $f, g_1$ and $g_2$, the symbol $(f, g_1)$ is a non-zero element of $\text{kur}^2 (L(Q(f, g_1 g_2)))/\mathbb{C}) = 0$. 

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- For appropriate choice of $f, g_1$ and $g_2$, the symbol $(f, g_1)$ is a non-zero element of $k^u_r(L(Q(f, g_1 g_2))/\mathbb{C})$.
- Recall $k^u_r(L(Q(f, g_1 g_2))/\mathbb{C}) = 0$. 

Aravind Asok (UCLA)  Rational connectivity and $\mathbb{A}^1$-connectivity
Application to rationality problems II

Example (Non-rational quadric bundles I)

Colliot-Thélène-Ojanguren ’89; Take $L = \mathbb{C}(x_1, x_2, x_3)$.

Let $f_1, f_2, g_1, g_2$ in $L^*$, and consider the quadric $Q(f_1, f_2, g_1 g_2)$. For appropriate choice of $f_1, f_2, g_1$ and $g_2$, the symbol $(f_1, f_2, g_1)$ is a non-zero element of $\text{ker}_3(L(Q(f_1, f_2, g_1 g_2)) / \mathbb{C})$.

Furthermore $\text{ker}_i(L(Q(f_1, f_2, g_1 g_2)) / \mathbb{C}) = 0$ for $i = 1, 2$.

Example (Non-rational quadric bundles II)

Peyre ’93: generalized these constructions of unramified elements and non-rational quadrics using $\text{ker}_4$.

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Application to rationality problems II

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Furthermore $k_4^{ur}(L(Q(f_1, f_2, g_1 g_2))/\mathbb{C}) = 0$ for $i = 1, 2$.
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- Furthermore $k^u_i(L(Q(f_1, f_2, g_1 g_2))/\mathbb{C}) = 0$ for $i = 1, 2$.

Example (Non-rational quadric bundles II)

- Peyre ’93: generalized these constructions of unramified elements and non-rational quadrics using $k^u_4$. 

Application to rationality problems II
Application to rationality problems III

Theorem (More non-rational quadric bundles)

Set $L = \mathbb{C}(x_1, ..., x_n)$. For every integer $n > 0$, there exist elements $(f_1, ..., f_n)$ in $L^*$ such that the quadric $Q(f_1, ..., f_n)$ is non-rational, and where non-rationality is detected by existence of a non-trivial element of $k_{ur}(L(Q(f_1, ..., f_n)) / \mathbb{C})$. Furthermore $k_{ur}(L(Q(f_1, ..., f_n)) / \mathbb{C}) = 0$ for $1 \leq i < n$. 

Aravind Asok (UCLA)
Application to rationality problems III

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Aravind Asok (UCLA)  
Rational connectivity and $\mathbb{A}^1$-connectivity
What lessons have we learned?

All the quadric bundles in question are rationally connected. As \( n \) increases, intuitively one imagines the examples we have constructed as being "closer and closer" to rational varieties. One might imagine heirarchies of "higher rational connectivity" to make these notions precise (cf. A.J. de Jong-J. Starr). Concretely, as \( n \) increases, "some kind of mod 2 cohomology" vanishes in higher and higher degrees.
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Connectedness in $\mathbb{A}^1$-homotopy theory

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**Definition**

A smooth variety $X$ over a field $F$ is $\mathbb{A}^1$-chain connected if for every finitely generated, separable extension $L/K$, any two $L$-points of $X$ can be connected by a chain of copies of the affine line.
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More generally, there is a notion of $\pi_0^{\mathbb{A}^1}$ that underlies this notion of connectedness (defined using the $\mathbb{A}^1$-homotopy category). For smooth proper $X$: think of chain-connected components.
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**Theorem**

*If $X/F$ is $\mathbb{A}^1$-chain connected, then all “unramified invariants” of $X$ are “trivial” (i.e., isomorphic to the value of the unramified invariant on the base-field).*
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**Theorem**

If $X/F$ is $\mathbb{A}^1$-chain connected, then all “unramified invariants” of $X$ are “trivial” (i.e., isomorphic to the value of the unramified invariant on the base-field).

**Corollary**

If $X/F$ has a “non-trivial” unramified invariant, then $F$ is not stably rational.
Basic principle: $\pi^{A^1}_0(X)$ controls all unramified invariants of $X$. 
Homological interpretation

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- Topological fact: if $A$ is a discrete abelian group, and $M$ is a manifold, then continuous maps $M \to A$ are in bijection with group homomorphisms $H_0(M) \to A$. 

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- Let $A$ be an unramified invariant (thought of as a functor on field extensions).
- Concrete incarnation (Morel): Unramified invariants on $X$ correspond bijectively with morphisms of functors $H_0^{\mathbb{A}^1}(X) \to A$. 
The upshot

Rost's study of the small Pfister quadrics (i.e., construction of the Rost motive) should allow one to understand the homomorphisms $\text{HA}_1^0(\mathbb{Q}(f_1,...,f_n)) \rightarrow k_{\text{ur}}$. For the rationality problem: Completely understand $\text{HA}_1^0(X)$ (even in the case of conics or small Pfister quadrics, this is open as far as I know).

There are many natural generalizations: e.g., so-called norm varieties can be used construct other examples of "bundles" that are rationally connected yet not $\text{A}_1$-connected.

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Thank you!

See http://www.math.ucla.edu/~asok for more information