Rational points up to stable $\mathbb{A}^1$-homotopy
joint with C. H"asemeyer and F. Morel

Aravind Asok (USC)

August 15, 2010
1 Zero cycles of degree 1
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2 Grothendieck-Witt groups
Outline

1. Zero cycles of degree 1
2. Grothendieck-Witt groups
3. Stable $\mathbb{A}^1$-homology
Throughout the talk we consider algebraic varieties over a field $k$ assumed to have characteristic 0.
Throughout the talk we consider algebraic varieties over a field $k$ assumed to have characteristic $0$.

All algebraic varieties will be assumed smooth (geometrically integral).
0-cycles

- Given an algebraic variety $X/k$, $CH_0(X)$ is the Chow group of 0-cycles modulo rational equivalence on $X$. 
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By definition, there is a surjection

$$\bigoplus_{x \in X(0)} \mathbb{Z} \longrightarrow CH_0(X)$$

where

- we write $X(0)$ for the set of dimension 0 points of $X$. 
Given an algebraic variety $X/k$, $CH_0(X)$ is the Chow group of 0-cycles modulo rational equivalence on $X$.

By definition, we can realize $CH_0(X)$ as the cokernel

$$\bigoplus_{x \in X(1)} \kappa_x^* \xrightarrow{\partial} \bigoplus_{x \in X(0)} \mathbb{Z} \rightarrow CH_0(X)$$

where

- we write $X(i)$ for the set of dimension $i$ points of $X$,
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- $\kappa_x$ (resp. $\kappa_x^*$) is the (group of units in the) residue field at $x$, and
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- $\kappa_x$ (resp. $\kappa_x^*$) is the (group of units in the) residue field at $x$, and
- $\partial$ is the “divisor” map.
The degree homomorphism

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that induces the usual degree homomorphism

$$\text{deg} : CH_0(X) \to \mathbb{Z}.$$
0-cycles of degree 1

Definition
An algebraic variety $X/k$ has a 0-cycle of degree 1 if the degree homomorphism

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An algebraic variety \( X/k \) has a 0-cycle of degree 1 if the degree homomorphism

\[ \text{deg} : \text{CH}_0(X) \rightarrow \mathbb{Z} \]

is surjective; a 0-cycle of degree 1 is a lift \( x \in \text{CH}_0(X) \) of \( 1 \in \mathbb{Z} \).
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- If $X$ has a $k$-rational point, then $X$ has a 0-cycle of degree 1.
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is surjective; a 0-cycle of degree 1 is a lift $x \in CH_0(X)$ of $1 \in \mathbb{Z}$.

- If $X$ has a $k$-rational point, then $X$ has a 0-cycle of degree 1.
- The converse is false, there exist varieties with points over extensions of coprime degrees, but that have no rational point.
Basic question

Question

Given an algebraic variety $X/k$, if $X$ has a 0-cycle of degree 1, can one give additional hypotheses guaranteeing that $X$ actually has a rational point?
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Given an algebraic variety $X/k$, if $X$ has a 0-cycle of degree 1, can one give additional hypotheses guaranteeing that $X$ actually has a rational point?

- There are a number of classical conjectures related to the above question, e.g., the Cassels-Swinnerton-Dyer conjecture.
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- If $(W, \psi)$ is another symmetric bilinear form over $L$. write $\langle a \rangle$ for the 1-dimensional vector space $L$ equipped with symmetric bilinear form $(x, x') = axx'$. 
Symmetric bilinear forms

Suppose \( L \) is a field. Let \((V, \varphi)\) be a symmetric bilinear form over \( L \) (we only consider non-degenerate forms).

- If \((W, \psi)\) is another symmetric bilinear form over \( L \)
  - we can equip \( V \oplus W \) with the symmetric bilinear form \( \varphi \oplus \psi \)
    defined by \( \varphi \oplus \psi((v \oplus w), (v' \oplus w')) = \varphi(v, v') + \psi(w, w') \),
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Suppose $L$ is a field. Let $(V, \varphi)$ be a symmetric bilinear form over $L$ (we only consider non-degenerate forms).

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  - we can equip $V \oplus W$ with the symmetric bilinear form $\varphi \oplus \psi$ defined by $\varphi \oplus \psi((v \oplus w), (v' \oplus w')) = \varphi(v, v') + \psi(w, w')$, and
  - we can equip $V \otimes W$ with the symmetric bilinear form $\varphi \otimes \psi$ defined by $\varphi \otimes \psi((v \otimes w), (v' \otimes w')) = \varphi(v, v') \cdot \psi(w, w')$. 

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  - we can equip $V \otimes W$ with the symmetric bilinear form $\varphi \otimes \psi$ defined by $\varphi \otimes \psi((v \otimes w), (v' \otimes w')) = \varphi(v, v') \cdot \psi(w, w')$.

- Write $\langle a \rangle$ for the 1-dimensional vector space $L$ equipped with symmetric bilinear form $(x, x') = axx'$. 
Functoriality of forms

If $k \hookrightarrow L$ is an arbitrary extension.

- Given a symmetric bilinear form $(V, \varphi)$ over $k$, we can extend scalars to obtain a symmetric bilinear form $(V_L, \varphi_L)$ over $L$.  

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Functoriality of forms

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- Extension of scalars preserves direct sums and tensor products.
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Suppose $k \hookrightarrow L$ is a finite extension.

- Let $tr_{L/k} : L \rightarrow k$ be the corresponding trace map.
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Suppose $k \hookrightarrow L$ is a finite extension.

- Let $tr_{L/k} : L \rightarrow k$ be the corresponding trace map.
- We can view $L$ as a symmetric bilinear space over $k$ by means of the trace form $(x, y) \mapsto tr_{L/k}(xy)$. 
Functoriality of forms

If $k \hookrightarrow L$ is an arbitrary extension.

- Given a symmetric bilinear form $(V, \varphi)$ over $k$, we can extend scalars to obtain a symmetric bilinear form $(V_L, \varphi_L)$ over $L$.
- Extension of scalars preserves direct sums and tensor products.

Suppose $k \hookrightarrow L$ is a finite extension.

- Let $tr_{L/k} : L \rightarrow k$ be the corresponding trace map.
- We can view $L$ as a symmetric bilinear space over $k$ by means of the trace form $(x, y) \mapsto tr_{L/k}(xy)$.
- Or, given a symmetric bilinear form $(V, \varphi)$ over $L$, we can obtain a symmetric bilinear form over $k$ by viewing $V$ as a vector space over $k$ and composing $\varphi$ with $tr_{L/k}$.
Grothendieck-Witt groups

**Definition**

The *Grothendieck-Witt group of a field* $k$, denoted $GW(k)$, is the Grothendieck group of the monoid of isomorphism classes of symmetric bilinear forms over $k$ with operation determined by the direct sum.
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The *Grothendieck-Witt group of a field* $k$, denoted $GW(k)$, is the Grothendieck group of the monoid of isomorphism classes of symmetric bilinear forms over $k$ with operation determined by the direct sum.

- $GW(k)$ is a commutative ring with multiplication induced by tensor product and unit given by the class of the symmetric bilinear form $⟨1⟩$. 

Extension of scalars and “trace” induce corresponding maps of Grothendieck-Witt groups:

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Functoriality of Grothendieck-Witt groups

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Extension of scalars and “trace” induce corresponding maps of Grothendieck-Witt groups:

- an extension $k \hookrightarrow L$ induces a ring homomorphism $GW(k) \to GW(L)$,
- a finite extension $k \hookrightarrow L$ induces a “transfer” homomorphism $GW(L) \to GW(k)$,
- the transfer homomorphism is a map of $GW(k)$ modules, and
- there is a dimension homomorphism $\text{dim} : GW(k) \to \mathbb{Z}$. 
Quadratic obstructions

Taking the sum of the transfer homomorphisms (varying over the dimension 0 points of $X$) defines a map

\[
\tilde{\deg} : \bigoplus_{x \in X_{(0)}} GW(\kappa_x) \longrightarrow GW(k).
\]
Quadratic obstructions

Taking the sum of the transfer homomorphisms (varying over the dimension 0 points of $X$) defines a map

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Elements of the direct sum on the left are 0-cycles equipped with a symmetric bilinear form.
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- $GW(k)$-linear, and
Quadratic obstructions

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Elements of the direct sum on the left are 0-cycles equipped with a symmetric bilinear form. Forgetting the symmetric bilinear form gives a map to the groups we already considered. Furthermore, the homomorphism $\widetilde{\text{deg}}'$ is

- $GW(k)$-linear, and
- factors through a “nice” quotient.
Rational points up to stable $\mathbb{A}^1$-homotopy

Definition

If $X/k$ is an algebraic variety, say $X$ has a rational point up to stable $\mathbb{A}^1$-homotopy if the map $\tilde{\deg}$ is surjective; a rational point up to stable $\mathbb{A}^1$-homotopy is a choice of lift of $\langle 1 \rangle \in GW(k)$. 
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- If $X$ has a $k$-rational point, then $X$ has a rational point up to stable $\mathbb{A}^1$-homotopy.
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- If $X$ has a $k$-rational point, then $X$ has a rational point up to stable $\mathbb{A}^1$-homotopy.
- If $X$ has a rational point up to stable $\mathbb{A}^1$-homotopy, then $X$ has a 0-cycle of degree 1.
Main questions

Question

What are general hypotheses under which the converses to the above statements hold?
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*What are general hypotheses under which the converses to the above statements hold? In other words,*

1. *if* $X$ *has a 0-cycle of degree 1, when does* $X$ *have a rational point up to stable* $\mathbb{A}^1$-*homotopy?*
Main questions

Question

What are general hypotheses under which the converses to the above statements hold? In other words,

1. If $X$ has a 0-cycle of degree 1, when does $X$ have a rational point up to stable $\mathbb{A}^1$-homotopy?

2. If $X$ has a rational point up to stable $\mathbb{A}^1$-homotopy, when does $X$ have a rational point?
A partial answer

**Theorem**

Suppose $k$ is not formally real, i.e., $-1$ is a sum of squares in $k$. If $X$ has a 0-cycle of degree 1, then $X$ has a rational point up to stable $\mathbb{A}^1$-homotopy (but $X$ need not have a rational point).
A partial answer

**Theorem**

Suppose $k$ is not formally real, i.e., $-1$ is a sum of squares in $k$. If $X$ has a 0-cycle of degree 1, then $X$ has a rational point up to stable $\mathbb{A}^1$-homotopy (but $X$ need not have a rational point).

The proof of the theorem is not very difficult: it uses the fact (Pfister ’66) that if $k$ is not formally real, then the ordinary Witt group $W(k)$ is a local ring.
Theorem

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If $k$ is formally real it seems reasonable to expect that there are varieties with a 0-cycle of degree 1, but no rational point up to stable $\mathbb{A}^1$-homotopy.
The notion of rational point up to stable $\mathbb{A}^1$-homotopy has good formal properties.
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**Theorem**

1. If $X$ and $X'$ are birationally equivalent smooth proper varieties over $k$, then $X$ has a rational point up to stable $\mathbb{A}^1$-homotopy if and only if $X'$ does.
The notion of rational point up to stable $\mathbb{A}^1$-homotopy has good formal properties.

**Theorem**

1. If $X$ and $X'$ are birationally equivalent smooth proper varieties over $k$, then $X$ has a rational point up to stable $\mathbb{A}^1$-homotopy if and only if $X'$ does.

2. Existence of a rational point up to stable $\mathbb{A}^1$-homotopy is a stable $\mathbb{A}^1$-homotopy invariant.
The $\mathbb{A}^1$-derived category

That the indices on $CH_0$ are written as subscripts is supposed to suggest that we are thinking about homology (the degree map is the pushforward to a point in Chow theory).
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- Start with the category $Sm_k$. 

The $\mathbb{A}^1$-derived category
Conventions
Zero cycles of degree 1
Grothendieck-Witt groups
Stable $\mathbb{A}^1$-homology
Milnor-Witt K-theory

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- Write $\mathcal{A}b_k$ for the category of Nisnevich sheaves of abelian groups on $Sm_k$. 

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The $\mathbb{A}^1$-derived category

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- Start with the category $Sm_k$.
- Write $\mathcal{Ab}_k$ for the category of Nisnevich sheaves of abelian groups on $Sm_k$.
- Write $D_-(k)$ for the corresponding derived category.
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- Start with the category $Sm_k$.
- Write $\mathcal{A}b_k$ for the category of Nisnevich sheaves of abelian groups on $Sm_k$.
- Write $D_-(k)$ for the corresponding derived category.
- Write $D_{\mathbb{A}^1}^{\text{eff}}(k)$ for the category obtained from $D_-(k)$ by “formally inverting” the maps $\mathbb{Z}(X \times \mathbb{A}^1) \to \mathbb{Z}(X)$. 

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- Start with the category $Sm_k$.
- Write $\mathcal{A}b_k$ for the category of Nisnevich sheaves of abelian groups on $Sm_k$.
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- Write $D^\text{eff}_{\mathbb{A}^1}(k)$ for the category obtained from $D_-(k)$ by “formally inverting” the maps $\mathbb{Z}(X \times \mathbb{A}^1) \to \mathbb{Z}(X)$.

The resulting category is an analog of Voevodsky’s triangulated category of effective motivic complexes “without transfers.”
The $\mathbb{A}^1$-chain complex

If $X$ is a variety, let $C_{A^1}^*(X)$ be the class of $\mathbb{Z}(X)$ in $D^{\text{eff}}_{A^1}(X)$. 
The $\mathbb{A}^1$-chain complex

- If $X$ is a variety, let $C^\mathbb{A}^1_*(X)$ be the class of $\mathbb{Z}(X)$ in $D^\text{eff}_{\mathbb{A}^1}(X)$.
- Denote by $\mathbb{Z}<1>$ the (co-)chain complex $C^\mathbb{A}^1_*(\mathbb{P}^1)[-2]$; this “is” a Tate twist.
The $\mathbb{A}^1$-chain complex

- If $X$ is a variety, let $C^\mathbb{A}^1_*(X)$ be the class of $\mathbb{Z}(X)$ in $D^\text{eff}_{\mathbb{A}^1}(X)$.
- Denote by $\mathbb{Z}\langle 1 \rangle$ the (co-)chain complex $C^\mathbb{A}^1_*(\mathbb{P}^1)[-2]$; this “is” a Tate twist.
- The $\mathbb{A}^1$-homology (sheaves) of $X$ are the homology sheaves of $C^\mathbb{A}^1_*(X)$. 
The $\mathbb{A}^1$-chain complex

- If $X$ is a variety, let $C_{\mathbb{A}^1}^*(X)$ be the class of $\mathbb{Z}(X)$ in $D_{\mathbb{A}^1}^{\text{eff}}(X)$.
- Denote by $\mathbb{Z}\langle 1 \rangle$ the (co-)chain complex $C_{\mathbb{A}^1}^*(\mathbb{P}^1)[-2]$; this “is” a Tate twist.
- The $\mathbb{A}^1$-homology (sheaves) of $X$ are the homology sheaves of $C_{\mathbb{A}^1}^*(X)$.
- The category $D_{\mathbb{A}^1}^{\text{eff}}(k)$ has many good formal properties, in particular a tensor product.
The stable $\mathbb{A}^1$-derived category

- If we want a good duality formalism, it turns out we have to invert the operation $\otimes \mathbb{Z}\langle 1 \rangle$. 
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Let $D_{\mathbb{A}^1}(k)$ be the category where $\mathbb{Z}\langle 1 \rangle$ is inverted.
The stable $\mathbb{A}^1$-derived category

- If we want a good duality formalism, it turns out we have to invert the operation $\otimes \mathbb{Z}\langle 1 \rangle$.
- Let $D_{\mathbb{A}^1}(k)$ be the category where $\mathbb{Z}\langle 1 \rangle$ is inverted.
- Formally, morphisms in the new category are colimits of the diagrams

$$\text{Hom}_{D_{\mathbb{A}^1}^{\text{eff}(k)}}(A, B) \to \text{Hom}_{D_{\mathbb{A}^1}^{\text{eff}(k)}}(A \otimes \mathbb{Z}\langle 1 \rangle, B \otimes \mathbb{Z}\langle 1 \rangle) \to \cdots$$

(just like Voevodsky’s triangulated category of motives).
Stable $\mathbb{A}^1$-homology

Definition

For a smooth variety $X$, we define

$$H_{0}^{\mathbb{A}^1}(X) := \text{Hom}_{\mathbb{D}_{\mathbb{A}^1}(k)}(\mathbb{Z}, C_{\ast}^{\mathbb{A}^1}(X)).$$
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**Definition**

For a smooth variety $X$, we define

$$H^s_{\mathbb{A}^1}(X) := \text{Hom}_{D_{\mathbb{A}^1}(k)}(\mathbb{Z}, C^*_{\mathbb{A}^1}(X)).$$

The sheaf $H^s_{\mathbb{A}^1}(X)$ is...
**Stable $\mathbb{A}^1$-homology**

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For a smooth variety $X$, we define

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The sheaf $H^s_{\mathbb{A}^1}(X)$ is

- $\mathbb{A}^1$-homotopy invariant by construction;
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- $\mathbb{A}^1$-homotopy invariant by construction;
- initial among (stable) strictly $\mathbb{A}^1$-homotopy invariant sheaves admitting a map from $X$;
- a birational invariant by combining the two observations above with Morel’s stable $\mathbb{A}^1$-connectivity theorem (via the Bloch-Ogus theory as formalized by Colliot-Thélène-Hoobler and Kahn).
The fact that the sheaf $H^0_{\mathbb{A}^1}(X)$ is initial means that it “controls all the unramified invariants of $X$.”
The fact that the sheaf $H_0^{s\mathbb{A}^1}(X)$ is initial means that it “controls all the unramified invariants of $X$.” For example,

- If we write $H^i_{\text{ét}}(\mu_n \otimes j)$ for the Nisnevich sheaf on $Sm_k$ associated with $U \mapsto H^i_{\text{ét}}(U, \mu_n \otimes j)$. 

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- then $H^i_\text{ur}(X, \mu_n \otimes j) = \text{Hom}(\mathbb{H}_0^{s\mathbb{A}^1}(X), \mathbb{H}_\text{ét}^i(\mu_n \otimes j))$. 

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- then $H^i_{\text{ur}}(X, \mu_n \otimes j) = \text{Hom}(H^{sA^1}_0(X), H^i_{\text{ét}}(\mu_n \otimes j))$.
- If we let $W$ denote the sheaf of unramified Witt groups,
The fact that the sheaf $H_0^{s\mathbb{A}^1}(X)$ is initial means that it “controls all the unramified invariants of $X$.” For example,

- If we write $H^i_{\text{ét}}(\mu_n \otimes j)$ for the Nisnevich sheaf on $Sm_k$ associated with $U \mapsto H^i_{\text{ét}}(U, \mu_n \otimes j)$.
- Then $H^i_{ur}(X, \mu_n \otimes j) = \text{Hom}(H_0^{s\mathbb{A}^1}(X), H^i_{\text{ét}}(\mu_n \otimes j))$.
- If we let $W$ denote the sheaf of unramified Witt groups,
- then $W_{ur}(X) = \text{Hom}(H_0^{s\mathbb{A}^1}(X), W)$. 
Further properties

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Rational points up to stable $\mathbb{A}^1$-homotopy
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- These sheaves are covariantly functorial for maps of schemes.
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- The hint that stable homology is linked to symmetric bilinear forms stems from a fundamental computation of Morel: for any extension $L/k$, $H_{0\mathbb{A}^1}(\text{Spec } k)(L) = GW(L)$. 
Further properties

- These sheaves are covariantly functorial for maps of schemes.
- If $X$ is furthermore proper, the relationship between the $\mathbb{A}^1$-derived category and Voevodsky’s category gives a map $H_{0}^{sA^1}(X) \rightarrow CH_0(X)$.
- The hint that stable homology is linked to symmetric bilinear forms stems from a fundamental computation of Morel: for any extension $L/k$, $H_{0}^{sA^1}(\text{Spec } k)(L) = GW(L)$.
- Our fundamental computation gives a description of the sheaf $H_{0}^{sA^1}(X)$ for smooth proper varieties amalgamating the data from “zero cycles” and “symmetric bilinear forms” above.
The main computation

**Theorem**

For any smooth proper variety $X$ over a field $k$ (char. 0) and any extension $L/k$, there is a canonical isomorphism

$$H^0_{s\mathbb{A}^1}(X)(L) \sim \tilde{C}H_0(X_L),$$

where $\tilde{C}H_0(X_L)$ is the group of “enhanced” 0-cycles on $X$. 
The main computation

Theorem

For any smooth proper variety $X$ over a field $k$ (char. 0) and any extension $L/k$, there is a canonical isomorphism

$$\mathbf{H}^{s\mathbb{A}^1}_0(X)(L) \xrightarrow{\sim} \widetilde{CH}_0(X_L),$$

where $\widetilde{CH}_0(X_L)$ is the group of “enhanced” 0-cycles on $X$. (Up to a small twist) the group $\widetilde{CH}_0(X_L)$ is the quotient of $\bigoplus_{x \in X(0)} GW(\kappa_x)$ we discussed earlier, and the pushforward to a point is induced by the map $\widetilde{\deg}'$. 

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Duality and the proof

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Duality and the proof

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- The “small twist” comes from the fact that essentially none of the varieties we are considering is “orientable” and thus when using duality we have to twist by an “orientation local system.”
- The explicit presentation arises from an unwinding of the Gersten resolution.
Thank you!

See http://www-bcf.usc.edu/~asok for more information
Given a field \( k \), Hopkins and Morel defined a graded ring \( K_{*}^{MW}(k) \) by generators and relations.
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**Definition**

Let $T_*^{MW}(k)$ be the free graded associative algebra with a generator $[u]$ of degree +1 for each $u \in F^*$ and with $\eta$ of degree $-1$. If $h = \eta[-1] + 2$, then $\eta \cdot h = 0$. 

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Given a field $k$, Hopkins and Morel defined a graded ring $K_{MW}^*(k)$ by generators and relations.

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Given a field $k$, Hopkins and Morel defined a graded ring $K^*_{MW}(k)$ by generators and relations.

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For a variety $X/k$, there is a “generalized divisor” map

$$\bigoplus_{x \in X_{(1)}} K_{1}^{MW}(\kappa_{x}) \xrightarrow{\partial} \bigoplus_{x \in X_{(0)}} K_{0}^{MW}(\kappa_{x})$$
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When $X$ has trivial canonical bundle, the cokernel of this map is the group $\widehat{CH}_{0}(X)$.