(1)(a) Since $E(Y_k \mid F_{k-1}) \geq Y_{k-1}$ we have for $k > n$ that

$$E(Y_k \mid F_n) = E(E(Y_k \mid F_{k-1}) \mid F_n) \geq E(Y_{k-1} \mid F_n),$$

so $\{E(Y_k \mid F_n), k \geq n\}$ is monotone nondecreasing a.s., so $\lim_k E(Y_k \mid F_n)$ exists a.s. (possibly $+\infty$.) Let $M = \sup_k EY_k < \infty$. By Monotone Convergence, $E(\lim_k E(Y_k \mid F_n)) = \lim_k E(\sum_k E(Y_k \mid F_n)) = \lim_k EY_k \leq M$, we must have $\lim_k E(Y_k \mid F_n) < \infty$ a.s.

(b) Let $Y_n$ be the size of the $n$th generation in a branching process with mean family size $\mu = 1$ (with family size not a.s. equal to 1.) We showed in lecture that $\mu$ is a martingale, so $\{\mu_i, n \geq 0\}$ is a martingale. Let $Y_n = \sum X_i$ for $0 \leq i \leq n$, so $\{Y_n \mid F_n\}$ is a martingale. Let $\phi, \Omega$ be the size of the $n$th generation in a branching process with mean family size $\mu = 1$ (with family size not a.s. equal to 1.) We showed in lecture that $\mu$ is a martingale, so $\{\mu_i, n \geq 0\}$ is a martingale. Let $Y_n = \sum X_i$ for $0 \leq i \leq n$, so $\{Y_n \mid F_n\}$ is a martingale.

(c) From (a), $EZ_n \leq M$ for all $n$, so $Z_n$ is $L^1$-bounded. By the Monotone Convergence Theorem for conditional expectation (Chapter 4 (1.1c) p. 223) we have

$$E(Z_{n+1} \mid F_n) = E\left(\lim_k E(Y_k \mid F_{n+1}) \mid F_n\right) = \lim_k E(Y_k \mid F_{n+1}) \mid F_n) = \lim_k E(Y_k \mid F_n) = Z_n,$$

so $\{Z_n\}$ is a martingale.

(d) Let $Y_k' = X^+_k$ and $Y_k'' = X^-_k$. These are both $L^1$-bounded submartingales (since the positive and negative parts are convex functions), so by (c), $Z_n' = \lim_k E(Y_k' \mid F_n)$ and $Z_n'' = \lim_k E(Y_k'' \mid F_n)$ are both nonnegative $L^1$-bounded martingales. Also

$$Z_n' - Z_n'' = \lim_k E(Y_k' - Y_k'' \mid F_n) = \lim_k E(Y_k \mid F_n) = Z_n.$$

(2)(a) Let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$. By the strong Markov property for $\{X_n\}$, since $\{Y_n = i_n, \ldots, Y_0 = i_0\} \in \mathcal{F}_{T_n}$,

$$P(Y_{n+1} = j \mid Y_n = i_n, \ldots, Y_0 = i_0) = P(X_{T_i} \circ \theta_{T_n} = j \mid Y_n = i_n, \ldots, Y_0 = i_0)$$

$$= P(X_{T_i} = j \mid X_0 = i_n)$$

$$= P(X_{T_i} \circ \theta_{T_n} = j \mid Y_n = i_n)$$

$$= P(Y_{n+1} = j \mid X_n = i_n),$$

so $\{Y_n\}$ is a Markov chain.

(b) Let $N_n(x)$ be the number of visits to $x$ by $\{X_n\}$ up to time $n$, and $L_n(x) = |\{k \leq n : Y_k = x\}|$. Let $S_x = \inf\{n \geq 1 : Y_n = x\}$. Note that $k/T_k$ is the fraction of time $\{X_n\}$ spends in $A$ up to time $T_k$, so

$$\frac{T_k}{k} \rightarrow \frac{1}{\pi(A)} \quad \text{as } k \rightarrow \infty, \text{ a.s.}$$
Hence for \( x \in A \),
\[
\frac{L_k(x)}{k} = \frac{N_{T_k}(x)}{T_k} \frac{T_k}{k} \to \frac{\pi(x)}{\pi(A)} \quad \text{as } k \to \infty, \text{ a.s.}
\]

It follows from Theorem 5.1 (p. 308) that we must have \( E_x S_x = \pi(A)/\pi(x) < \infty \), so \( \{Y_n\} \) is positive recurrent.

(3) Let \( X_n \) be the number of umbrellas at the professor’s location when he starts his \( n \)th trip. For \( 1 \leq i \leq N \) we have
\[
P(X_{n+1} = N - i + 1 \mid X_n = i) = p, \quad P(X_{n+1} = N - i \mid X_n = i) = 1 - p,
\]
and for \( i = 0 \), we have \( P(X_{n+1} = N \mid X_n = 0) = 1 \). The same is true if we also condition on the values of \( X_0, \ldots, X_{n-1} \), so \( \{X_n\} \) is a Markov chain. Since the chain is irreducible with finite \( S \), a stationary distribution \( \pi \) exists. Each time the chain visits state 0, it is independently raining or not when the professor departs, so the long-run fraction of trips when he gets wet is \( \pi(0)p \).

For \( 1 \leq i \leq N - 1 \) we have from the stationarity condition that
\[
\pi(N - i)(1 - p) + \pi(N - i + 1)p = \pi(i),
\]
while \( \pi(N)(1 - p) = \pi(0) \) and \( \pi(0) + \pi(1)p = \pi(N) \). Combining the last two we get
\[
\pi(N)(1 - p) + \pi(1)p = \pi(N)
\]
which simplifies to \( \pi(N - j) = \pi(j) \) which by assumption is equal to \( \pi(1) \). We claim that in fact we have \( \pi(1) = \ldots = \pi(N) \). To see this, suppose that we have
\[
\pi(1) = \pi(2) = \ldots = \pi(j) = \pi(N - j + 1) = \ldots = \pi(N)
\]
(that is, the first \( j \) values and the last \( j \) values are all equal), for some \( j < N/2 \). This is known for \( j = 1 \), from the above. Taking \( i = j \) in (1) we get \( \pi(j) = \pi(N - j)(1 - p) + \pi(N - j + 1)p = \pi(N - j)(1 - p) + \pi(j)p \) which simplifies to \( \pi(N - j) = \pi(j) \) which by assumption is equal to \( \pi(1) \). Similarly we then get \( \pi(j + 1) = \pi(1) \) so (2) is true for \( j + 1 \) in place of \( j \). Therefore by induction (2) is true for all \( j \leq N/2 \), proving the claim.

It follows that \( 1 = \pi(0) + \pi(1) + \ldots + \pi(N) = \pi(N)(1 - p + N) \) so \( \pi(N) = \frac{1}{N + 1 - p} \) and then \( \pi(0) = (1 - p)\pi(N) = \frac{1 - p}{N + 1 - p} \), so the long run fraction of wet trips is \( \pi(0) = \frac{p(1 - p)}{N + 1 - p} \).

(4) Starting from 0, in order to hit \( j \) before returning to 0 the first step must be to +1 and then one must reach \( j \) from 1 before hitting 0. Thus letting \( T_x = \min\{n \geq 1 : X_n = x\} \),
\[
P_0(N_j \geq 1) = P_0(T_j < T_0) = pP_1(T_j < T_0) = p \frac{\frac{q}{p} - 1}{\left(\frac{q}{p}\right)^j - 1}.
\]

Similarly, starting from \( j \), in order to hit 0 before returning to \( j \) the first step must be \(-1\) and then one must reach 0 from \( j - 1 \) before hitting \( j \), so
\[
P_j(T_0 < T_j) = qP_{j-1}(T_0 < T_j) = q \left( 1 - \frac{\left(\frac{q}{p}\right)^{j-1} - 1}{\left(\frac{q}{p}\right)^j - 1} \right) = q \frac{\left(\frac{q}{p}\right)^j - 1}{\left(\frac{q}{p}\right)^j - 1}.
\]

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By the Markov property, for \( k \geq 1 \), letting \( T_x^{(k)} \) be the time of the \( k \)th visit to \( x \) (excluding visits at time 0) before the first return to 0, we have

\[
P_0(N_j \geq k + 1 \mid N_j \geq k) = \sum_{l=k}^{\infty} P_0(T_j^{(k+1)} < \infty \mid T_j^{(k)} = l) P_0(T_j^{(k)} = l \mid T_j^{(k)} < \infty) = \sum_{l=k}^{\infty} P_j(T_j < T_0) P_0(T_j^{(k)} = l \mid T_j^{(k)} < \infty) = P_j(T_j < T_0).
\]

This shows that given \( N_j \geq 1 \), \( N_j - 1 \) has a geometric distribution with parameter \( P_j(T_0 < T_j) \), so

\[
E_0 N_j = E_0(N_j 1_{\{N_j \geq 1\}}) = E_0(N_j \mid N_j \geq 1) P_0(N_j \geq 1) = \frac{P_0(N_j \geq 1)}{P_j(T_0 < T_j)} = \frac{p}{q} \frac{q - 1}{p^j - (q/p)^{j-1}}
\]

which simplifies to \((p/q)^j\).

(5) Let \( \mathcal{F}_n = \sigma(X_0, \ldots, X_n) \). Let \( x > K \). By the Markov property, on the event \( \{ X_{n\land\tau} = x \} = \{ X_n = x \} \cap \{ \tau > n \} \) we have \( Y_{(n+1)\land\tau} = X_{n+1} + (n+1)\epsilon \) so

\[
E(Y_{(n+1)\land\tau} \mid \mathcal{F}_n) = E(X_{n+1} \mid X_n = x) + (n+1)\epsilon
= E_x X_1 + (n+1)\epsilon
\leq x - \epsilon + (n+1)\epsilon
= Y_{n\land\tau}.
\]

If \( x \leq K \) then on the event \( \{ X_{n\land\tau} = x \} = \{ X_\tau = x \} \cap \{ \tau \leq n \} \) we have \( Y_{(n+1)\land\tau} = Y_\tau = x + \tau \epsilon \) so

\[
E(Y_{(n+1)\land\tau} \mid \mathcal{F}_n) = x + \epsilon E(\tau \mid \mathcal{F}_n) = x + \epsilon \tau = X_\tau + \epsilon \tau = Y_{n\land\tau}.
\]
Thus \( \{ Y_{n\land\tau} \} \) is a positive supermartingale. Since \( X_n \geq 0 \), and using Monotone Convergence,

\[
x = E_x Y_0 \geq E_x Y_{n\land\tau} \geq \epsilon E_x (n \land \tau) \to \epsilon E_x \tau \quad \text{as } n \to \infty.
\]
Therefore \( E_x \tau \leq x/\epsilon \).