(1) The mean of $I(f, t) = \int_0^t f(s) \, dW_s$ is 0 since it’s a martingale and $I(f, 0) = 0$, so the variance is $EI(f, t)^2 = \int_0^t E(f(s))^2 \, ds$. In particular:

(a) For $f(s) = |W_s|^{1/2}$, we have $Ef(s)^2 = E|W_s| = s^{1/2}EW_1 = (2s)^{1/2}$, and the variance is $\left(\frac{2}{3}\right)^{1/2} \int_0^t s^{1/2} \, ds = \frac{2}{3} \left(\frac{2}{3}\right)^{1/2} t^{3/2}$.

(b) For $f(s) = (W_s + s)^2$ we have

$$Ef(s)^2 = E(W_s^4 + 4sW_s^3 + 6s^2W_s^2 + 4s^3W_s + s^4) = s^2EW_1^4 + 6s^3EW_1^2 + s^4 = 3s^2 + 6s^3 + s^4,$$

and the variance is $\int_0^t (3s^2 + 6s^3 + s^4) \, ds = t^3 + \frac{3}{2}t^4 + \frac{1}{5}t^5$.

(2)(a) Let $H(x) = e^x$ so $Z_t = H(Y_t)$. Then $dY_t = f(t) \, dW_t - \frac{1}{2} f(t)^2 \, dt$ so by Ito’s Lemma,

$$dZ_t = H'(Y_t) \, dY_t + \frac{1}{2} H''(Y_t) f(t)^2 \, dt = e^{Y_t} \left( f(t) dW_t - \frac{1}{2} f(t)^2 \, dt \right) + \frac{1}{2} e^{Y_t} f(t)^2 \, dt = e^{Y_t} f(t) \, dW_t.$$

(b) Yes, because by (a), $Z_t = I(g, t)$ for $g(t) = e^{Y_t} f(t)$. (Actually we need an assumption here to ensure that $\int_0^t E g(s)^2 \, ds < \infty$, for example that $f$ is bounded.)

(3)(a) Let $Y_t = Z_t - \int_0^t (u(s) + \frac{1}{2} v(s)^2) \, ds$, so $M_t = e^{Y_t}$ and

$$dY_t = dZ_t - u(t) \, dt - \frac{1}{2} v(t)^2 \, dt = v(t) \, dW_t - \frac{1}{2} v(t)^2 \, dt.$$

By Ito’s Lemma with $H(x) = H'(x) = H''(x) = e^x$ and $M_t = H(Y_t)$,

$$dM_t = H'(Y_t) \, dY_t + \frac{1}{2} H''(Y_t) v(t)^2 \, dt$$

$$= M_t \left( v(t) \, dW_t - \frac{1}{2} v(t)^2 \, dt + \frac{1}{2} v(t)^2 \, dt \right)$$

$$= M_t v(t) \, dW_t,$$

so $M_t$ is a martingale.

(b) From (a) we have $h(t) = M_t v(t)$. In the case of constant $u, v$ and $Z_0 = 0$,

$$Z_t = \int_0^t u \, ds + \int_0^t v \, dW_s = ut + vW_t,$$

so

$$Y_t = Z_t - \int_0^t (u + \frac{1}{2} v^2) \, ds = Z_t - ut - \frac{1}{2} v^2 t = vW_t - \frac{1}{2} v^2 t,$$

and then

$$h(t) = vM_t = ve^{vW_t - \frac{1}{2} v^2 t}.$$
(4) Let \( h(x) = x^3 \) and \( X_t = W_t^3 \). By Ito’s Lemma,
\[
dX_t = 3W_t^2 \, dW_t + \frac{1}{2} (6W_t) \, dt
\]
so
\[
W_t^3 = X_t - X_0 = \int_0^t 3W_s^2 \, dW_s + \int_0^t 3W_s \, ds,
\]
or equivalently,
\[
\int_0^t W_s^2 \, dW_s = \frac{1}{3} W_t^3 - \int_0^t W_s \, ds.
\]

(5)(a) By Ito’s Lemma,
\[
\begin{align*}
\frac{dh(Y_t)}{dt} &= h'(Y_t) \left( m(Y_t) \, dt + \sqrt{Y_t(1-Y_t)} \, dW_t \right) + \frac{1}{2} h''(Y_t) Y_t(1-Y_t) \, dt \\
&= \sqrt{Y_t(1-Y_t)} \, dW_t + \left( h'(Y_t)m(Y_t) + \frac{1}{2}h''(Y_t)Y_t(1-Y_t) \right) dt.
\end{align*}
\]
Thus \( h(Y_t) \) is a martingale provided \( h'(x)m(x) + \frac{1}{2} h''(x)x(1-x) = 0 \) for all \( x \), or equivalently,
\[
\frac{d}{dx} \log h'(x) = \frac{h''(x)}{h'(x)} = \frac{-2m(x)}{x(1-x)} = \frac{x - \frac{1}{2}}{x(1-x)} = \frac{1}{2} \left( \frac{1}{1-x} - \frac{1}{x} \right).
\]
Integrating, we get
\[
\log h'(x) = \frac{1}{2} (-\log(1-x) - \log x) + c = \frac{1}{2} \log \frac{1}{x(1-x)} + c,
\]
and we can choose \( c = 0 \) since we only need one solution, so
\[
h'(x) = \frac{1}{\sqrt{x(1-x)}}.
\]
Integrating again using the substitution \( u = \sin^2 \theta \) we get
\[
h(x) = \int_0^x \frac{1}{\sqrt{u(1-u)}} \, du = \int_0^{\sin^{-1} \sqrt{x}} 2 \, d\theta = 2 \sin^{-1} \sqrt{x}.
\]

(b) Note that as \( x \) increases from 0 to 1, \( h(x) \) increases from 0 to \( \pi \). Let \( T = \inf \{ t > 0 : Y_t \in \{ \frac{1}{2}, \frac{3}{4} \} \} = \inf \{ t > 0 : h(Y_t) \in \{ h(\frac{1}{2}), h(\frac{3}{4}) \} \} \). Since \( h(Y_{t\wedge T}) \) is a bounded martingale, for the process starting from \( Y_0 = 5/8 \) we have
\[
h \left( \frac{5}{8} \right) = E h(Y_0)
= E h(Y_T)
= h \left( \frac{3}{4} \right) P \left( Y_T = \frac{3}{4} \right) + h \left( \frac{1}{2} \right) \left( 1 - P \left( Y_T = \frac{3}{4} \right) \right)
\]
so

\[ P \left( Y_T = \frac{3}{4} \right) = \frac{h \left( \frac{5}{8} \right) - h \left( \frac{1}{2} \right)}{h \left( \frac{3}{4} \right) - h \left( \frac{1}{2} \right)} = .485. \]

(c) For \(0 < a < y < b < 1\), similarly to (b) we have

\[ P_y(T_a < T_b) = \frac{h(b) - h(y)}{h(b) - h(a)}. \]

Now since \( T_b < \infty \) and \( Y_t \) is continuous, if \( T_a < T_b \) for all \( a \in (0, y) \) then \( T_0 < T_b \). Thus for \( a_n \downarrow 0 \) we have \( \{T_0 < T_b\} = \bigcap_{n=1}^{\infty} \{T_{a_n} < T_b\} \) and hence

\[ P_y(T_0 < T_b) = \lim_{n} P_y(T_{a_n} < T_b) = \lim_{n} \frac{h(b) - h(y)}{h(b) - h(a_n)} = \frac{h(b) - h(y)}{h(b) - h(0)} < \infty. \]