(1) Let $W_t$ be Brownian motion. Find the variance of:
(a) $\int_0^t |W_s|^{1/2} \, dW_s$,
(b) $\int_0^t (W_s + s)^2 \, dW_s$.

(2) Let $f(t)$ be an adapted process, let $W_t$ be Brownian Motion and let
\[ Y_t = \int_0^t f(s) \, dW_s - \frac{1}{2} \int_0^t f(s)^2 \, ds, \quad Z_t = e^{Y_t}. \]
(a) Calculate the stochastic differential $dZ_t$.
(b) Is $Z_t$ a martingale? How do you know?

(3)(a) Let $W_t$ be Brownian motion. Suppose $dZ_t = u(t) \, dt + v(t) \, dW_t$, with $u, v \in L^2$, and
\[ M_t = \exp \left( Z_t - \int_0^t \left( u(s) + \frac{1}{2} v(s)^2 \right) \, ds \right). \]
Show that $dM_t = h(t) dW_t$ for some $h$, and hence by “Lemma 1” from lecture, $M_t$ is a martingale. You may assume $u, v, Z$ are bounded so you don’t have to worry about integrability.
(b) Calculate $h(t)$ explicitly (in terms of $W_t$) if $u$ and $v$ are constants and $Z_0 = 0$.

(4) Let $W_t$ be Brownian motion. Show that
\[ \int_0^t W_s^2 \, dW_s = \frac{1}{3} W_t^3 - \int_0^t W_s \, ds. \]

(5) Consider the following stochastic differential equation which arises as a model of evolution of gene frequencies ($Y_t$ = frequency of the gene at time $t$):
\[ dY_t = m(Y_t) \, dt + \sqrt{Y_t(1-Y_t)} \, dW_t, \]
where $\{W_t\}$ is Brownian motion and $m(u) = \frac{1}{4} - \frac{1}{2} u$.
(a) Find a function $h$ such that $h(Y_t)$ is a martingale. (It’s OK if $h(y) \to \infty$ as $y \to 0$ or as $y \to 1$, but $h(Y_t)$ should be well-defined while $Y_t \in (0, 1)$.)
(b) Suppose the diffusion starts at $Y_0 = 5/8$. Find $P(\{Y_t\}$ hits $3/4$ before it hits $1/2$).
(c) Let $T_0 = \inf\{t : Y_t = 0\}$. Show that $P^y(T_0 < \infty) > 0$. Here $P^y$ denotes probability when $\{Y_t\}$ starts from $Y_0 = y$. 

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For problem (5) you may assume without proof that for \( \{Y_t\} \) the hitting time of every \( a \in (0, 1) \) is always finite.

HINTS:

(2)(b) Don’t do this by calculation of \( E(Z_t \mid \mathcal{F}_s) \).

(3)(b) First express \( M_t \) as \( \varphi(W_t) \) for some \( \varphi \).

(5)(a) See problem 3a. (b) Use (a).
(c) Consider the probability of hitting a before b for \( 0 < a < b < 1 \), and take an appropriate limit.