Chapter 7:

(1.3) Let \(\xi\) be \(N(0, t)\), so \(\Delta_{m,n}\) has the distribution of \(2^{-n/2}\xi\). We have \(E\xi^2 = t, E(\xi^2-t)^2 = 2t^2\), so the r.v.'s \(\Delta_{m,n}^2 - \frac{t}{2^n}\) are i.i.d. with mean 0. Therefore

\[
E\left(\sum_{m=1}^{2^n} \Delta_{m,n}^2 - t\right)^2 = E\left(\sum_{m=1}^{2^n} \left(\Delta_{m,n}^2 - \frac{t}{2^n}\right)\right)^2
\]

\[
= \sum_{m=1}^{2^n} E\left(\Delta_{m,n}^2 - \frac{t}{2^n}\right)^2
\]

\[
= 2^n \cdot \frac{1}{2^n} E(\xi^2-t)^2
\]

\[
= \frac{2t^2}{2^n}.
\]

Combining this with Chebyshev’s Inequality we obtain

\[
\sum_{n=1}^{\infty} P\left(\left|\sum_{m=1}^{2^n} \Delta_{m,n}^2 - t\right| > \epsilon\right) \leq \sum_{n=1}^{\infty} \frac{2t^2}{2^n \epsilon^2} < \infty,
\]

so by Borel-Cantelli,

\[
P\left(\left|\sum_{m=1}^{2^n} \Delta_{m,n}^2 - t\right| > \epsilon \text{ i.o. in } n\right) = 0.
\]

Since \(\epsilon\) is arbitrary this means \(\sum_{m=1}^{2^n} \Delta_{m,n}^2 \rightarrow t\) a.s.

(2.1) We have \(R = T_0 \circ \theta_1 + 1\) so by the Markov property,

\[
P_x(R > 1 + t \mid \mathcal{F}_1) = P_x(T_0 \circ \theta_1 > t \mid \mathcal{F}_1) = P_{X_1}(T_0 > t).
\]

Hence

\[
P_x(R > 1 + t) = E_x P_{X_1}(T_0 > t) = \int_{\mathbb{R}} p_1(x, y) P_y(T_0 > t) \, dy,
\]

since \(X_1\) has the density \(p_1(x, y)\) (as a function of \(y\)) under \(P_x\).

(2.3) By Theorem 2.6 (and its symmetric analog for \(B_t < 0\)), \(B_t\) goes back and forth between positive and negative values infinitely often a.s. as \(t \searrow 0\). This means that almost surely, there exist \(s_1 > t_1 > s_2 > t_2 \geq \ldots\) with \(B(s_n) = 0, B(t_n) > 0\). Therefore there is
Thus there is a local maximum in each interval $(s_{n+1}, s_n)$, occurring wherever $\sup_{t \in [s_{n+1}, s_n]} B_t$ occurs. By the Markov property, for all $a < b$, $P(B_t$ has a local maximum in $(a, b)) = P(B_t - B_a$ has a local maximum in $(a, b)) = P(B_t - B_0$ has a local maximum in $(0, b - a)) = 1$.

(2.4)(i) Let $X = \lim \sup_{t \to 0} \frac{B(t)}{\sqrt{t}}$. Then $X \in \mathcal{F}_0^+$ so by Theorem 2.5, $P(X < x) = 0$ or $1$, for each $x$. The only such d.f. that of a constant r.v., so there exists $c$ such that $P(X = c) = 1$.

(ii) We have $\frac{B(t)}{\sqrt{t}} \sim N(0, 1)$ so given $M < \infty$ we have $P\left(\frac{B(t)}{\sqrt{t}} > M\right) \not\to 0$ as $t \to \infty$. Hence for all $M$,

$$P(X \geq M) \geq \lim_{\epsilon \to 0} P\left(\frac{B(t)}{\sqrt{t}} > M \text{ for some } t \leq \epsilon\right) \geq \lim_{\epsilon \to 0} P\left(\frac{B(\epsilon)}{\sqrt{\epsilon}} > M\right) \neq 0,$$

so by (i), $P(X \geq M) = 1$ for all $M$, which means $X = \infty$ a.s.

(3.2) For all $t$ we have

$$\{S \wedge T < t\} = \{S < t\} \cup \{T < t\} \in \mathcal{F}_t,$$

$$\{S \vee T < t\} = \{S < t\} \cap \{T < t\} \in \mathcal{F}_t,$$

and $S + T < t \iff$ there exists $q \in \mathbb{Q} \cap [0, t]$ with $S < q, T < t - q$, which means

$$\{S + T < t\} = \cup_{q \in \mathbb{Q} \cap [0, t]} \{S < q\} \cap \{T < t - q\} \in \mathcal{F}_t.$$

The last sentence in the problem follows from the fact that $T \equiv t$ is a stopping time.

(3.3)(i) $\{\sup_n T_n \leq t\} = \cap_n \{T_n \leq t\} \in \mathcal{F}_t$.

(ii) $\{\inf_n T_n \geq t\} = \cap_n \{T_n \geq t\} \in \mathcal{F}_t$.

(iii) $\{\limsup_n T_n < t\} = \{\lim_n (\sup_{m \geq n} T_m) < t\} = \cup_n \{\sup_{m \geq n} T_m < t\} \in \mathcal{F}_t$ by (i). Here the last equality uses the fact that $\sup_{m \geq n} T_m$ is decreasing in $n$.

(iv) $\{\liminf_n T_n > t\} = \{\lim_n (\inf_{m \geq n} T_m) > t\} = \cup_n \{\inf_{m \geq n} T_m > t\} \in \mathcal{F}_t$ by (ii). Here the last equality uses the fact that $\inf_{m \geq n} T_m$ is increasing in $n$.

(3.7) Let $S$ be a stopping time and $S_n = \lfloor \frac{2^n S}{2^n} \rfloor + 1$, so $S_n \searrow S$. For Borel $A$ and $t > 0$,

$$\{B(S_n) \in A\} \cap \{S_n \leq t\} = \cup_{1 \leq m \leq 2^n} \{S_n = \frac{m}{2^n}\} \cap \{B(\frac{m}{2^n}) \in A\}. \quad (2)$$

Now for $m \leq 2^n t$, $\{S_n = \frac{m}{2^n}\} = \{\frac{m-1}{2^n} \leq S < \frac{m}{2^n}\} \in \mathcal{F}_{m/2^n} \subset \mathcal{F}_t$ and $\{B(\frac{m}{2^n}) \in A\} \in \mathcal{F}_{m/2^n} \subset \mathcal{F}_t$, so the right side of (2) in in $\mathcal{F}_t$. This shows that $\{B(S_n) \in A\} \in \mathcal{F}_t$ since $A$ is arbitrary. For $m \geq n$, we have $S_m \leq S_n$, so by Theorem 3.5, $B(S_m) \in \mathcal{F}_{S_m} \subset \mathcal{F}_{S_n}$. Hence $B(S) = \lim_m B(S_m) \in \mathcal{F}_{S_n}$ for all $n$. Then by Theorem 3.6, $B(S) \in \cap_n \mathcal{F}_{S_n} = \mathcal{F}_S$. 

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