Chapter 5:

(1.2) 1 → 2 in two steps means 1 → 3 → 2, so \( p^2(1, 2) = p(1, 3)p(3, 2) = (.9)(.4) = .36. \)

2 → 3 in 3 steps means 2 → 1 → 3 or 2 → 2 → 1 → 3 or 2 → 1 → 1 → 3, so

\[
p^3(2, 3) = (.7)(.9)(.6) + (.3)(.7)(.9) + (.7)(.1)(.9) = .63.
\]

(1.3) We claim that for all \( \mu \) and all \( n \geq 0, \)

\[
P_\mu(X_n = 0) = \frac{\beta}{\alpha + \beta} + (1 - \alpha - \beta)^n \left\{ \mu(0) - \frac{\beta}{\alpha + \beta} \right\}.
\]

Equivalently, subtracting from 1,

\[
P_\mu(X_n = 1) = \frac{\alpha}{\alpha + \beta} - (1 - \alpha - \beta)^n \left\{ \mu(0) - \frac{\beta}{\alpha + \beta} \right\}.
\]

For \( n = 0 \) the first equality says \( P_\mu(X_n = 0) = \mu(0) \) which is true by definition. Suppose the claim is true for some \( n \). Then

\[
P_\mu(X_{n+1} = 0) = P(X_{n+1} = 0 \mid X_n = 0)P_\mu(X_n = 0) + P(X_{N=1} = 1 \mid X_n = 1)P_\mu(X_n = 1)
\]

\[
= (1 - \alpha) \left( \frac{\beta}{\alpha + \beta} + (1 - \alpha - \beta)^n \left\{ \mu(0) - \frac{\beta}{\alpha + \beta} \right\} \right)
\]

\[
+ \beta \left( \frac{\alpha}{\alpha + \beta} - (1 - \alpha - \beta)^n \left\{ \mu(0) - \frac{\beta}{\alpha + \beta} \right\} \right)
\]

\[
= \frac{\beta}{\alpha + \beta} + (1 - \alpha - \beta)^{n+1} \left\{ \mu(0) - \frac{\beta}{\alpha + \beta} \right\},
\]

so the claim is true for \( n + 1 \).

(1.5) Since just one son and daughter from each generation are mated, and any other siblings are irrelevant, we may assume that each generation contains exactly one male and one female. The son effectively chooses a gene at random from each parent, independently. The daughter does the same, independently of the son. If, for example, the parents are \( Aa, Aa \) then the son and daughter are (without regard to order):
AA, AA with probability $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$,
AA, Aa with probability $2 \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{4}$,
AA, aa with probability $2 \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{8}$,
Aa, Aa with probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$,
Aa, aa with probability $2 \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{4}$,
aa, aa with probability $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$.

By similar calculations we get the rest of the transition matrix (for the states in the order above), with the 4th row given by the probabilities above:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\frac{1}{16} & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{16} \\
0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

(1.6) Let $F_n = \sigma(\xi_1, ..., \xi_n)$. Now

$$X_{n+1} = \begin{cases} 
X_n + 1 & \text{if } \xi_{n+1} \notin \{\xi_1, ..., \xi_n\}, \\
X_n & \text{if } \xi_{n+1} \in \{\xi_1, ..., \xi_n\}
\end{cases}
$$

so

$$P(X_{n+1} = X_n + 1 \mid F_n) = 1 - \frac{X_n}{n}, \quad P(X_{n+1} = X_n \mid F_n) = \frac{X_n}{n},$$

Since this depends only on $X_n$, $\{X_n\}$ is a Markov chain, and $p(k, k+1) = 1 - \frac{k}{N}, p(k, k) = \frac{k}{N}$, with all other $p(k, l) = 0$.

(1.7) Suppose that for some $n, k$, we have $X_{n-1} = k - 1$, and $X_n = k$. Then $k$ is a new maximum value achieved by $\{S_j\}$ at time $n$, meaning $S_n = k$. Conditionally on all this, a new maximum equal to $k + 1$ occurs at time $n + 1$ provided $\xi_{n+1} = 1$. This shows that

$$P(X_{n+1} = k + 1 \mid X_{n-1} = k - 1, X_n = k) = P(\xi_{n+1} = 1 \mid X_{n-1} = k - 1, X_n = k)$$

$$= P(\xi_{n+1} = 1)$$

$$= \frac{1}{2}.$$

In contrast, suppose $X_{n-3} = k - 1, X_{n-2} = k, X_{n-1} = k, X_n = k$. Then $k$ was a new maximum value achieved at time $n - 2$, meaning $S_{n-2} = k$ and then $S_{n-1} = k - 1$ and $S_n$ is either $k - 2$ or $k$, with probability $1/2$ each. If $S_n = k - 2$ then there is 0 probability of a
new maximum at time \( n + 1 \), while if \( S_n = k \) then as in (1) there is probability 1/2 of a new maximum. Thus we have

\[
P(X_{n+1} = k + 1 \mid X_{n-3} = k - 1, X_{n-2} = X_{n-1} = X_n = k) \\
= P(X_{n+1} = k + 1 \mid X_{n-3} = k - 1, X_{n-2} = X_{n-1} = X_n = k, S_n = k) \\
\cdot P(S_n = k \mid X_{n-3} = k - 1, X_{n-2} = X_{n-1} = X_n = k) \\
+ P(X_{n+1} = k + 1 \mid X_{n-3} = k - 1, X_{n-2} = X_{n-1} = X_n = k, S_n = k - 2) \\
\cdot P(S_n = k - 2 \mid X_{n-3} = k - 1, X_{n-2} = X_{n-1} = X_n = k)
\]

\[
= \frac{1}{2} \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} \\
= \frac{1}{4}.
\]

If the process \( \{X_j\} \) were Markov, then this and (1) would both be equal to \( P(X_{n+1} = k + 1 \mid X_n = k) \). The fact they are not equal thus shows that \( \{X_j\} \) is not Markov.

(1.8) Suppose \( i_1, \ldots, i_n \) are each equal to \( \pm 1 \), including \( k \) 1’s and \( n - k \) -1’s. Then (using the Beta distribution formula to calculate the integral),

\[
P(X_1 = i_1, \ldots, X_n = i_n, X_{n+1} = 1) = E[P(X_1 = i_1, \ldots, X_n = i_n, X_{n+1} = 1 \mid \theta)] \\
= E(\theta^{k+1}(1 - \theta)^{n-k}) \\
= \int_{0}^{1} \theta^{k+1}(1 - \theta)^{n-k} d\theta \\
= \frac{(k + 1)(n - k)!}{(n + 2)!},
\]

and similarly

\[
P(X_1 = i_1, \ldots, X_n = i_n) = \frac{k!(n - k)!}{(n + 1)!},
\]

so

\[
P(X_{n+1} = 1 \mid X_1 = i_1, \ldots, X_n = i_n) = \frac{P(X_1 = i_1, \ldots, X_n = i_n, X_{n+1} = 1)}{P(X_1 = i_1, \ldots, X_n = i_n)} = \frac{k + 1}{n + 2}.
\]  \( \text{(2)} \)

Since \( S_n = k - (n - k) = 2k - n \), we have \( k = \frac{S_n + n}{2} \), meaning the left side of (2) depends only on \( S_n \); therefore (2) is equal to \( P(X_{n+1} = 1 \mid S_n = 2k - n) \), and \( \{S_n\} \) is Markov. Thus (2) says

\[
P(S_{n+1} = 2k - n + 1 \mid S_n = 2k - n) = P(X_{n+1} = 1 \mid S_n = 2k - n) = \frac{k + 1}{n + 2}.
\]
or letting $j = 2k - n$,  
\[ P(S_{n+1} = j + 1 \mid S_n = j) = \frac{n + j + 2}{2(n + 2)} \cdot \]

Since this transition probability depends on $n$, the Markov chain $\{S_n\}$ is time-inhomogeneous.

(A)(i) Clearly $\varphi(0) = 1$ and $\lim_{t \to -\infty} \varphi(t) = 0$. Consider $t \geq 0$; then
\[ \varphi'(t) = \frac{-q'(t)}{(1 + q(t))^2} \]

Calculating $\varphi''$ by differentiating this gets messy, so instead we will consider the increasing/decreasing and positive/negative properties of the numerator and denominator. Letting $\psi(t) = \log \left( \frac{c + 1}{t} \right)$,
\[ q'(t) = t\alpha \left( \log \left( \frac{c + 1}{t} \right) \right)^{\alpha - 1} \left( \frac{-1}{c + 1} \right) + \left( \log \left( \frac{c + 1}{t} \right) \right)^\alpha = \psi(t)^\alpha - \frac{\alpha - 1}{ct + 1} \psi(t)^{\alpha - 1} . \]

Note $\psi(t) > 0$ for all $t > 0$, since $c > 1$. Using $\psi'(t) = -\frac{1}{t(c + 1)}$ we get
\[ q''(t) = \psi(t)^{\alpha - 2} \left( \frac{\alpha - 1}{t(ct + 1)} + \alpha \psi(t) \left( \frac{c}{ct + 1} - \frac{1}{t} \right) \right) , \]

which is negative since both terms in the parentheses are negative. Thus $q'$ is decreasing, and since $\lim_{t \to -\infty} q'(t) = (\log c)^\alpha > 0$, $q'(t)$ is also positive for all $t > 0$, i.e. $q$ is increasing. Hence in the above formula for $\varphi'$, the numerator is increasing and negative, while the denominator is increasing and positive, so $\varphi'$ is increasing and negative. Therefore $\varphi$ is convex. By the Polya Criterion (3.10 in Chapter 2, p. 102), $\varphi$ is a characteristic function.

(ii) The problem should say show $E|\xi_i| = \infty$, not $E\xi_i = \infty$. This follows from the fact that a finite mean would imply (by Dominated Convergence) that $\varphi$ is differentiable, but here $\varphi$ is not differentiable at 0 since $\varphi$ is an even function satisfying $(1 - \varphi(t))/t \to \infty$ as $t \to 0$.

To prove recurrence we use Theorem 2.9 of Chapter 3 (p. 189). Note $\varphi$ is real-valued.
We have (using the change of variable $u = \log \frac{2}{t}$):

\[
\int_{-\delta}^{\delta} \frac{1}{1 - \varphi(t)} \, dt = \int_{-\delta}^{\delta} \left(1 + \frac{1}{q(t)}\right) \, dt
\]

\[
= 2\delta + 2 \int_{0}^{\delta} \frac{1}{t \left(\log \left(c + \frac{1}{t}\right)\right)} \, dt
\]

\[
\geq 2 \int_{0}^{\delta} \frac{1}{t \left(\log \frac{2}{t}\right)} \, dt
\]

\[
= 2 \int_{2/\delta}^{\infty} \frac{1}{u^\alpha} \, du
\]

\[
= \infty.
\]

Here in the inequality we assumed that $\delta < 1/c$, so $c + 1/t \leq 2/t$ for all $t \in (0, \delta]$. It follows from Theorem 2.9 of Chapter 3 that the random walk is recurrent for all $\alpha \leq 1$ and $c > 1$.

(B) Let $T_k$ be the $k$th index $n \geq 1$ for which $S_n \leq 0, S_{n+1} > 0$, with $T_k = \infty$ if there is no such index. Then $T_k$ is a stopping time, so by the Strong Markov property, on the event $\{T_k < \infty\}$,

\[
P_0(T_{k+1} < \infty \mid \mathcal{F}_{T_k}) = P_0(T_1 \circ \theta_{T_k} < \infty \mid S_{T_k}) = P_{S_{T_k}}(T_1 < \infty).
\]

Since $S_{T_k} > 0$, we have

\[
P_{S_{T_k}}(T_1 = \infty) \geq P_{S_{T_k}}(S_n > S_{T_k} \text{ for all } n > T_k) = P_0(S_n > 0 \text{ for all } n \geq 1) = \epsilon > 0.
\]

(This defines $\epsilon$.) Hence $P_0(T_{k+1} < \infty \mid \mathcal{F}_{T_k}) \leq 1 - \epsilon$, which implies $P_0(T_{k+1} < \infty \mid T_k < \infty) \leq 1 - \epsilon$, and then, inductively, $P_0(T_{k+1} < \infty) \leq (1 - \epsilon)kP_0(T_1 < \infty) \leq (1 - \epsilon)^{k+1}$. Thus

\[
P\left(\sum_n 1_{\{s_n \leq 0, s_{n+1} > 0\}} \geq k\right) = P(T_k < \infty) \leq (1 - \epsilon)^k,
\]

so

\[
\sum_n P(S_n \leq 0, S_{n+1} > 0) = E\left(\sum_n 1_{\{s_n \leq 0, s_{n+1} > 0\}}\right) < \infty.
\]