Lemma B.1 Let $R$ be defined by (26) and $m = \frac{1}{2} \max_{a} \left| u_{S}^{\prime}(a) \right| > 0$, and for each $r^{R} \in R$ define
\[
\Delta_{S} = \frac{\langle q^{S}, r^{R}x \rangle}{\langle q^{S}, r^{R} \rangle} - \langle p^{R}, x \rangle,
\]
and define $l_{r^{R}}(\varepsilon)$ as
\[
l_{r^{R}}(\varepsilon) = \frac{\langle \varepsilon, r^{R} \rangle}{\Delta_{S}}.
\]
For any $\varepsilon$ and $r^{R} \in R$ such that
\[
l_{r^{R}}(\varepsilon) < -m \text{ and } \Delta_{S} > 0, \text{ with } p^{S} + \varepsilon \in \Delta(\Theta),
\]
there exists a signal $\pi$ with the following properties: (i) Some realization of $\pi$ induces in the sender the belief $p^{S} + \varepsilon$ and (ii) $\pi$ increases the expected utility of the sender when the receiver’s associated likelihood ratio is $r^{R}$.

Proof: The function $l_{r^{R}}(\varepsilon)$ has an immediate interpretation as a measure of disagreement: the numerator $\langle \varepsilon, r^{R} \rangle$ is the difference in the probability that the receiver and sender attach to a signal realization inducing a posterior $q_{S} = p_{S} + \varepsilon$ on the sender, divided by the probability that the sender ascribes to such signal realization, while the denominator is the change in the receiver’s action when the sender changes her belief to $q_{S}$. We first show that if some $\varepsilon$ satisfies (40), then the value of information control is positive. Consider $V_{S}$ defined in (9), which in this case can be written as
\[
V_{S}(q^{S}) = u_{S} \left( \frac{\langle q^{S}, r^{R}x \rangle}{\langle q^{S}, r^{R} \rangle} \right),
\]
with gradient at $p^{S}$
\[
\nabla V_{S}(p^{S}) = u_{S}^{\prime}(\langle p^{R}, x \rangle) \left( r^{R}x - \langle p^{R}, x \rangle r^{R} \right).
\]
By Corollary 1, the value of information control is positive if and only if there exists $\varepsilon$, with $p^{S} + \varepsilon \in \Delta(\Theta)$, such that
\[
\langle \nabla V_{S}(p^{S}), \varepsilon \rangle < V_{S}(p^{S} + \varepsilon) - V_{S}(p^{S}).
\]
We now show that an $\varepsilon$ satisfying (40) also satisfies (41). Since
\[
\int_{\langle p^{R}, x \rangle}^{t} \int_{\langle p^{R}, x \rangle}^{t} u_{S}^{\prime}(\tau) \, d\tau \, dt,
\]
we can rewrite (41) as

$$u'_S(\langle p^R, x \rangle) \langle \varepsilon, r^R \rangle \Delta_S < \int_{\langle p^R, x \rangle}^{\langle q^S, r^R \rangle} \int_{\langle p^R, x \rangle}^{t} u''_S(\tau)d\tau dt.$$ 

By the mean value theorem, we have

$$\int_{\langle p^R, x \rangle}^{\langle q^S, r^R \rangle} \int_{\langle p^R, x \rangle}^{t} u''_S(\tau)d\tau dt \geq -\max |u''_S(a)| \int_{\langle p^R, x \rangle}^{\langle q^S, r^R \rangle} \int_{\langle p^R, x \rangle}^{t} d\tau dt = -\frac{1}{2} \max |u''_S(a)| \Delta^2_S.$$ 

Moreover, if $\varepsilon$ satisfies (40) then it also satisfies

$$\langle \varepsilon, r^R \rangle \min u'_S(a) < -\frac{1}{2} \max |u''_S(a)| \Delta_S,$$

implying that $\varepsilon$ also satisfies (41) since

$$u'_S(\langle p^R, x \rangle) \langle \varepsilon, r^R \rangle \Delta_S < \langle \varepsilon, r^R \rangle \Delta_S \min u'_S(a) < -\frac{1}{2} \max |u''_S(a)| \Delta^2_S \leq \int_{\langle p^R, x \rangle}^{\langle q^S, r^R \rangle} \int_{\langle p^R, x \rangle}^{t} u''_S(\tau)d\tau dt.$$ 

For each $\varepsilon$ satisfying (40), we now construct a signal that improves the sender's expected utility and that has a realization that induces belief $p^S + \varepsilon$ in the sender. Let $\nu$ be the excess of the right hand side over the left hand side in (41),

$$\nu = V_S(p^S + \varepsilon) - V_S(p^S) - \langle \nabla V_S(p^S), \varepsilon \rangle > 0. \quad (42)$$

Consider the signal $\pi(\varepsilon, \delta)$ with $Z = \{\varepsilon^+, \varepsilon^-\}$, such that $\Pr_S[z = \varepsilon^+] = \delta$ and if $z = \varepsilon^+$ then the sender’s posterior is $p^S + \varepsilon$. A taylor series expansion of $V_S(q^S)$ yields

$$V_S(q^S) = V_S(p^S) + \langle \nabla V_S(p^S), q^S - p^S \rangle + L (q^S - p^S), \text{ with } \lim_{t \to 0} \frac{L(t(q^S - p^S))}{t} = 0. \quad (43)$$

Then the sender’s gain from signal $\pi(\varepsilon, \delta)$ is

$$\Delta_{\pi(\varepsilon, \delta)} = \delta (V_S(p^S + \varepsilon) - V_S(p^S)) + (1 - \delta) \left(V_S(p^S - \frac{\delta}{1 - \delta} \varepsilon) - V_S(p^S)\right)$$

$$= \delta (\nu + \langle \nabla V_S(p^S), \varepsilon \rangle) - \delta \langle \nabla V_S(p^S), \varepsilon \rangle + L \left(-\frac{\delta}{1 - \delta} \varepsilon\right)$$

$$= \delta \left(\nu - (1 - \delta) \frac{L(-\delta \varepsilon/(1 - \delta))}{(-\delta/(1 - \delta))}\right).$$

The convergence to zero of the second term in the parenthesis when $\delta$ tends to zero and $\nu > 0$ guarantees the existence of $\delta > 0$ such that $\Delta_{\pi(\varepsilon, \delta)} > 0$. \[\square\]
Proof of Proposition 8: First, we introduce additional notation. With \( l_r(\varepsilon) \) defined as in (39), define the sets \( M(r^R) \) by

\[
M(r^R) = \{ \varepsilon : l_r(\varepsilon) < -m, \Delta_S > 0, p^S + \varepsilon \in \Delta(\Theta) \}.
\]

Note that \( r^S \) and \( x \) are negatively collinear if and only if \( r^R \) and \( r^Rx \) are positively collinear. That is, the condition on Proposition 5 could be instead stated in terms of collinearity of \( r^R \) and \( r^Rx \). Moreover, if \( r^R \) and \( r^Rx \) are not collinear then the restriction of \( l_r(\varepsilon) \) to \( \{ \varepsilon : \langle \varepsilon, 1 \rangle = 0 \} \) is surjective and thus the set \( M(r^R) \) is non-empty.

Define the function

\[
\Psi(\varepsilon, r^R) = \langle \varepsilon, r^R - mf^R \rangle + (\langle \varepsilon, r^R \rangle)^2, \quad \text{with} \quad f^R = r^Rx - \langle p^S, r^Rx \rangle,
\]

which characterizes \( M(r^R) \) since for \( \varepsilon \) such that \( p^S + \varepsilon \in \Delta(\Theta) \), \( \Psi(\varepsilon, r^R) \leq 0 \) and \( \langle \varepsilon, f^R \rangle \geq 0 \) if and only if \( \varepsilon \in M(r^R) \). Finally, let

\[
\gamma = 2 \left( 1 + m \left( \max \{x_\theta \} + \|x\| \right) + (4 + m \|x\|) \sup_{r^R \in R} \|r^R\| \right), \quad \text{(44)}
\]

\[
Z = \min_{\varepsilon \in \{ \varepsilon : p^S + \varepsilon \in \Delta(\Theta) \}, r^R \in R} \Psi(\varepsilon, r^R) \quad \text{s.t.} \quad \langle \varepsilon, r^R (x - \langle p^S, r^Rx \rangle) \rangle \leq 0, r^R \in R. \quad \text{(45)}
\]

Under the conditions of Proposition 8, \( Z < 0 \). Finally, define \( \beta \) in (27) as

\[
\beta = \frac{|Z|}{\gamma}. \quad \text{(46)}
\]

Our proof is structured in two steps that show (i) if \( \cap_{r^R \in R} M(r^R) \) is non-empty then following Lemma B.1 allows us to design a signal \( \pi \) that increases the sender’s expected utility for every receiver’s belief in the support of \( h(p^R|p^S) \), and (ii) under the conditions of Proposition 8, \( \cap_{r^R \in R} M(r^R) \neq \emptyset \).

Step (i) - Suppose that \( \varepsilon \in \cap_{r^R \in R} M(r^R) \). Consider \( v \) as defined by (42). As \( v \) is a continuous function of \( r^R \) in the compact set \( R \), it achieves a minimum \( \underline{v} = \min_{r^R \in R} v > 0 \). Then, define \( \hat{\delta} \) as

\[
\hat{\delta} = \min \left\{ \delta : \underline{v} + \frac{L(-\frac{\delta}{1-\beta}|\varepsilon|)}{\delta} \geq 0 \right\},
\]

with the function \( L \) given by (43). Now define the signal \( \pi(\varepsilon, \delta') \) as in the proof of Lemma B.1, i.e. \( Z = \{ \varepsilon^+, \varepsilon^- \}, q^S(\varepsilon^+) = p^S + \varepsilon \) and \( \Pr_S[z = \varepsilon^+] = \delta' \), and set \( \delta' = \hat{\delta} \). Then the sender’s gain from \( \pi(\varepsilon, \delta') \) is positive for any receiver’s prior in \( \text{Supp}(h(p^R|p^S)) \).
Step (ii) - Fix $p^R$ with associated likelihood ratio $r^R \in R$. For any $r^R \in R$ with $\eta = r^R - r^R$, we have

$$\Psi(\varepsilon, r^R) - \Psi(\varepsilon, r^R) = \left(1 + m \left\langle p^S, r^R x \right\rangle + 2 \left(\varepsilon, r^R + r^R\right)\right) \langle \varepsilon, \eta \rangle - m \langle \varepsilon, \eta x \rangle + m \left\langle p^S, \eta x \right\rangle \langle \varepsilon, r \rangle.$$  

The following bounds make use of the Cauchy-Schwartz inequality (in particular the implication that $|\langle \varepsilon, \eta x \rangle| \leq \|\varepsilon\| \|\eta\| \|x\|$, see Steele 2004\textsuperscript{21}) and the fact that $\|p^S\| \leq 1$ and $\|\varepsilon\| = \|q^S - p^S\| \leq 2$,

$$\left|1 + m \left\langle p^S, r^R x \right\rangle + 2 \left(\varepsilon, r^R + r^R\right)\right| \leq 1 + m \max_{r^R \in R} \|r^R\|,$$  

$m \langle \varepsilon, \eta x \rangle \leq m \|\varepsilon\| \|\eta\| \|x\| \leq 2m \|\eta\| \|x\|,$  

$m \left\langle p^S, \eta x \right\rangle \langle \varepsilon, r \rangle \leq 2m \|\eta\| \|x\| \sup_{r^R \in R} \|r^R\|.$

From these bounds, we then obtain the following estimate

$$\left|\Psi(\varepsilon, r^R) - \Psi(\varepsilon, r^R)\right| \leq \left|1 + m \left\langle p^S, r^R x \right\rangle + 2 \left(\varepsilon, r^R + r^R\right)\right| \|\varepsilon\| \|\eta\|$$  

$$+ m \langle \varepsilon, \eta x \rangle + m \left\langle p^S, \eta x \right\rangle \langle \varepsilon, r \rangle \leq 2 \left(1 + m \max_{r^R \in R} \|r^R\| \right) \|\eta\| + 2m \|x\| \|\eta\|$$  

$$+ 2m \|x\| \sup_{r^R \in R} \|r^R\| \|\eta\| = \gamma \|\eta\|,$$

where $\gamma$ is defined by (44). Selecting $\varepsilon'$ an $r^R$ that solve the program (45) and noting that $Z < 0$ we then have that for any $r^R \in R$,

$$\Psi(\varepsilon', r^R) = \Psi(\varepsilon', r^R) + \Psi(\varepsilon', r^R) - \Psi(\varepsilon', r^R) \leq Z + \gamma \|\eta\| \leq Z + |Z| = 0.$$

This implies that $\varepsilon' \in M(r^R)$ for all $r^R \in R$.\hfill $\blacksquare$