Research Statement

1 Introduction

My research interests lie in the broad area of algebraic geometry with a focus on semi-topological invariants associated to complex algebraic varieties. Complex algebraic varieties are defined as zero loci of a finite set of polynomial equations with complex coefficients.

Starting from early nineteen nineties, new invariants constructed topologically were associated to algebraic varieties defined over fields of characteristic zero. The search for a (co)homology theory for algebraic varieties, as envisioned by Beilinson, gave several constructions, produced by Bloch, Voevodsky, Suslin, Levine, Grayson, Friedlander and Lawson. Starting from the ground-breaking work of Lawson [35], Friedlander defined the Lawson homology for complex algebraic varieties as the homotopy groups of algebraic cycle spaces of a fixed dimension on X [13]. For a projective complex variety X, the topological group of r-cycles is defined as

$$Z_r(X) = (\amalg_{d\geq 0} C_{r,d}(X)^{an})^+$$

which is the naive completion of the monoid of effective algebraic cycles on X defined by using the Chow varieties $C_{r,d}(X)$. The topology on $Z_r(X)$ is given by the quotient topology and the empty cycle $0 \in C_r(X)$ is the natural base point of $Z_r(X)$. We write

$$L_rH_n(X) := \pi_{n-2r}(Z_r(X)) \tag{1.1}$$

and, intuitively, think of an element in this group as a "family of r-cycles parametrized by a (n-2r)-sphere" [17]. These groups contain both algebraic and topological information about the complex algebraic variety X. For example, the algebraic equivalence class of an algebraic r-cycle can be expressed as a connected component of the topological space of algebraic r-cycles $Z_r(X)$ [13]. In the same flavor, Dold-Thom theorem shows that "families of 0-cycles parametrized by a n-sphere" are the same as the topological classes in the singular homology of X^{an}. The s-map is a map that "measures" how close Lawson homology groups are to algebraic geometry or topology. We have the following sequence of maps

$$A_r(X) = L_rH_{2r}(X) \xrightarrow{s} L_{r-1}H_{2r}(X) \xrightarrow{s} .. \xrightarrow{s} L_1H_{2r}(X) \xrightarrow{s} H_{2r}(X^{an}). \tag{1.2}$$

The composition of the above s-maps gives the usual cycle map between the Chow group of algebraic cycles modulo algebraic equivalence, denoted $A_r(X)$, and the singular homology of the complex points of X, written $H_{2r}(X^{an})$ [17]. Encoded in the construction of the s-map is the celebrated suspension theorem for algebraic cycles proved by Lawson [35], the starting point of Lawson homology.

Lawson homology with finite coefficients of a smooth complex projective variety X is proved to be isomorphic, via a Poincare type duality [16], with the motivic cohomology with finite coefficients of X [22]. This isomorphism is the generalization of the fact that algebraic and rational equivalence for algebraic cycles coincide with finite coefficients. Through this isomorphism, the torsion of the Lawson homology groups can be studied using motivic cohomology tools. In particular, the Beilinson-Lichtenbaum conjecture may be used to identify, for certain indices, Lawson homology groups with finite coefficients with singular homology groups with finite coefficients. As shown in [45], the Beilinson-Lichtenbaum conjecture is equivalent to the Bloch-Kato conjecture, which has been proven by V. Voevodsky and M. Rost.

Friedlander and Mazur conjectured that Lawson homology groups should vanish for homological dimension greater than twice the dimension of a smooth complex variety in any weight, i.e.

Conjecture 1.1. Let X be a smooth complex projective variety. Then

$$L_rH_n(X) = \pi_{n-2r}(Z_r(X)) = 0$$

for any $n > 2dim(X)$.

For zero cycles, the Dold-Thom theorem says that

$$L_0H_n(X) = \pi_n(Z_0(X)) = H_n^{BM}(X, \mathbb{Z})$$

for any complex quasi-projective variety. This equals zero for any $n > 2dim(X)$.

Friedlander-Mazur conjecture was later included in a more general conjecture called Suslin's conjecture for complex varieties [14]. This is a extension to integer coefficients of the Beilinson-Lichtenbaum conjecture for motivic cohomology. Below is the formulation of the conjecture that is convenient for a case-by-case verification.

Conjecture 1.2. (Suslin's conjecture) The map

$$L_qH_n(X) \to H_n(X)$$

is an isomorphism for $n \geq d + q$ and a monomorphism for $n \geq d + q - 1$ for any smooth quasi-projective complex variety X of dimension d.

The Conjecture 1.1 and Conjecture 1.2 are very deep and very far reaching for the existent methods in the field.

In [47], Teh defined a reduced Lawson homology group for quasi-projective real varieties. Let X be a quasi-projective real variety. The Galois group $G = Gal(\mathbb{C}/\mathbb{R})$ acts on $\mathcal{Z}_q(X_{\mathbb{C}})$, the topological group of q-cycles on the complexification. Cycles on the real variety X correspond to cycles on $X_{\mathbb{C}}$ which are fixed by conjugation. Inside the topological group of $\mathcal{Z}_q(X_{\mathbb{C}})^G$ of cycles fixed by conjugation is the closed topological subgroup $\mathcal{Z}_q(X_{\mathbb{C}})^{av}$ of averaged cycles which are the cycles of the form $\alpha + \overline{\alpha}$. The space of reduced cycles on X is the quotient topological group

$$\mathcal{R}_q(X) = \frac{\mathcal{Z}_q(X_{\mathbb{C}})^G}{\mathcal{Z}_q(X_{\mathbb{C}})^{av}}. \tag{1.3}$$

Homotopy groups of some of the above abelian topological groups are related to classical topological invariants. For example for X a projective real variety it is known that the singular homology groups $\pi_*\mathcal{R}_0(X) = H_*(X(\mathbb{R}), \mathbb{Z}/2)$ [47] and $\pi_*\mathcal{Z}_0(X_{\mathbb{C}})^{av} = H_*(X_{\mathbb{C}}(\mathbb{C})/G, \mathbb{Z})$

[38]. Other homotopy groups are related to classical algebraic geometry invariants. For example $\pi_0(Z_r(X_{\mathbb{C}})^G)$ computes the group of algebraic cycles of dimension r on X modulo real algebraic equivalence [23] and consequently with $\mathbb{Z}/n$ coefficients equals the Chow group $CH_r(X) \otimes \mathbb{Z}/n$ (see [25]). In [47], Teh constructed a homology theory for projective real varieties, called reduced Lawson homology and defined

$$RL_rH_n(X) = \pi_{n-r}(R_r(X)) \tag{1.4}$$

for any $n \geq r$.

My work in [50], [49] was concerned with the study of Lawson homology of particular smooth complex projective varieties, such as rationally connected threefolds and fourfolds or generic cubic hypersurface of dimension eight, and showed that, in particular, they fulfilled Suslin's conjecture. More recently my work (with J. Heller) proved that the Friedlander-Mazur conjecture is valid for the reduced Lawson homology and current work is aimed at extending this line of research.

My projects for future research, detailed in section 3, include: (1) The study of topological groups of real algebraic cycles of a real variety, especially the reduced Lawson homology groups, (2) Extending a work of Suslin and Voevodsky and proving that Suslin's conjecture is equivalent with a Bloch-Kato type of conjecture for morphic cohomology; using this statement would allow us to prove Suslin's conjecture for smooth threefolds, which would give the strongest evidence available so far for the validity of Suslin's conjecture and (3) Studying possible extensions of Beilinson-Lichtenbaum conjecture, proved by V. Voevodsky and M. Rost, to the quaternionic morphic cohomology of a complex variety.

2 Previous work

2.1 Computations of Lawson homology and Semi-topological K-theory

In my paper [49], we studied the Lawson homology of smooth complex varieties of small dimension with zero cycles supported on a proper subvariety. One of the main motivation of this paper was to study Suslin's conjecture 1.2 and Friedlander-Mazur's conjecture 1.1 on concrete examples. We computed Lawson homology for these "degenerate" varieties of small dimension and confirmed, in particular, these two conjectures.

We studied the action of an algebraic cycle on the morphic cohomology of a smooth projective variety. Using this action, we showed that the Bloch and Srinavas decomposition of the diagonal cycle [3] induces isomorphisms and monomorphism for some of the rational generalized cycle maps from Lawson homology of X to singular homology of the analytic space of complex points of X. We also used Beilinson-Lichtenbaum conjecture to study the torsion of the kernel and cokernel of the generalized cycle maps. In addition we studied various other decompositions of the diagonal as for example the one given by Jannsen [28] and Laterveer [32].

The main result of the paper for threefold varieties is

Theorem 2.1. *Let X be a smooth projective complex threefold such that there is a proper subvariety $V \subset X$ with $CH_0(X \setminus V) = 0$. Then:*

$$K_i^{sst}(X) \simeq ku^{-i}(X^{an}), i \geq 1,$$

$$K_0^{sst}(X) \hookrightarrow ku^0(X^{an}).$$

Moreover if X is a rationally connected threefold then

$$K_i^{sst}(X) \simeq ku^{-i}(X^{an}), i \geq 0.$$

In particular, Suslin's conjecture is valid for such X.

In the case of a fourfold, we proved the following theorem

Theorem 2.2. *Let X be a smooth projective fourfold such that there is a proper subvariety $V \subset X$ of $dim(V) \leq 2$ with $CH_0(X \setminus V) = 0$. Then:*

$$K_i^{sst}(X) \simeq ku^{-i}(X^{an}), i \geq 3,$$

$$K_2^{sst}(X) \hookrightarrow ku^{-2}(X^{an}),$$
$$K_i^{sst}(X)_{\mathbb{Q}} \simeq ku^{-i}(X^{an})_{\mathbb{Q}}, i = 1, 2,$$
$$K_0^{sst}(X)_{\mathbb{Q}} \hookrightarrow ku^0(X^{an})_{\mathbb{Q}}.$$

In particular, Suslin's conjecture is valid for such X.

In particular, Theorem 2.1 and Theorem 2.2 generalize previous results of E.Friedlander, C.Haesemeyer and M.Walker [14] about the semi-topological K-theory of rational threefolds and fourfolds.

The methods of [49] are not very effective for rationally connected varieties of dimension higher than five. In my paper [50], we used a method introduced by J. Lewis in [39] to study the generalized cycle maps for rationally connected hypersurfaces of dimension higher than five. As an application we compute the rational semi-topological K-theory of a cubic sixfold and eightfold.

Theorem 2.3. *Let X be a generic cubic sixfold or eightfold. Then*

$$K_*^{sst}(X)_{\mathbb{Q}} \simeq ku_{\mathbb{Q}}^{-*}(X^{an})$$

for any $* \geq 1$ and

$$K_0^{sst}(X)_{\mathbb{Q}} \hookrightarrow ku_{\mathbb{Q}}^0(X^{an}).$$

2.2 Friedlander-Mazur Conjecture for Reduced Lawson Homology

In [25] we proved the analogue of Friedlander-Mazur conjecture for reduced Lawson homology. This work generalized results of Teh [46], and confirmed observations made on specific cases of real projective space and quaternionic projective space computed by Lam [31] and Lawson, Lima-Filho and Michelsohn [37]. The main object of study in [25] was the cycle map

$$RL_rH_n(X) \to H_n(X(\mathbb{R}), \mathbb{Z}/2) \tag{2.4}$$

defined by Teh [47]. Using a careful analysis of the Friedlander-Walker cycle map, we proved that the cycle map 2.4 is at least injective for $n > dim(X)$, which implies the following analogue of Friedlander-Mazur conjecture for reduced Lawson homology:

Theorem 2.5. *For any smooth quasi-projective variety* X

$$RL_rH_n(X) = \pi_{n-r}(R_r(X)) = 0$$

for any $n > dim(X)$.

This result may be seen as an application of the ground-breaking Milnor conjecture proved by Voevodsky [48]. According to Suslin and Voevodsky [45], Milnor conjecture implies the Beilinson-Lichtenbaum conjecture. Beilinson-Lichtenbaum conjecture states that the canonical map from motivic cohomology to etale cohomology

$$H^n(X, \mathbb{Z}/l(q)) \to H^n_{et}(X, \mu_l^{\otimes q}) \tag{2.6}$$

is an isomorphism for any $n \leq q$ and a monomorphism for any $n = q + 1$, for any smooth variety over a field of characteristic zero. We use Beilinson-Lichtenbaum map for varieties over $\mathbb{R}$ and $\mathbb{C}$. A technical result that we proved in [25] is:

Theorem 2.7. *Let* X *be a smooth quasi-projective real variety. The diagram commutes*

$$\begin{tikzcd}
L^qH\mathbb{R}^{q-k,q}(X;\mathbb{Z}/2) \arrow[d, "\Phi"'] & & \mathbb{H}^{2q-k,q}_{\mathcal{M}}(X;\mathbb{Z}/2) \arrow[ll, "\cong"'] \arrow[d, "cyc"] \\
H^{q-k,q}_{Br}(X_{\mathbb{C}}(\mathbb{C});\underline{\mathbb{Z}/2}) \arrow[r] & H^{2q-k}_{G}(X_{\mathbb{C}}(\mathbb{C});\mathbb{Z}/2) \arrow[r] & H^{2q-k}_{et}(X;\mu_2^{\otimes q}),
\end{tikzcd}$$

where $H^{p-q,q}_{Br}(X_{\mathbb{C}}(\mathbb{C});\underline{\mathbb{Z}/2})$ *denotes Bredon cohomology and* $H^p_G(X_{\mathbb{C}}(\mathbb{C});\mathbb{Z}/2)$ *denotes Borel cohomology.*

The bi-index Bredon cohomology is the RO(G)-graded equivariant cohomology theory for $G = \mathbb{Z}/2$. The map between Borel cohomology and etale cohomology is an isomorphism, by a result of Cox [5], which can also be reproved here as an isomorphism given by a natural map of complexes of Nisnevich sheaves. The map between Bredon cohomology and Borel cohomology is defined by the natural projection $X_{\mathbb{C}}(\mathbb{C}) \times E\mathbb{Z}/2 \to X_{\mathbb{C}}(\mathbb{C})$. In particular, the bottom horizontal maps are isomorphisms for any smooth projective real variety with no real points. In some indexes this is true for any smooth projective real variety. For example, if $q \leq p$, the left horizontal map coincides with the right horizontal map [25]. Therefore, we can use Milnor conjecture to study real Lawson homology of a real variety X. Theorem 2.7 suggests that we can use Bredon cohomology, instead of Borel cohomology, to detect motivic cohomology classes of smooth real varieties.

As an application of Theorem 2.5, we have the following vanishing theorem for average cycles.

Corollary 2.8. *Let X be a smooth projective real variety of dimension d. Then*

$$\pi_n \frac{\mathcal{Z}_p(X_{\mathbb{C}})^{av}}{2\mathcal{Z}_p(X_{\mathbb{C}})^{av}} = 0$$

for $n \geq 2d - 2p + 1$.

This theorem is expected to hold with integer coefficients, mainly because of the following description of Friedlander-Walker morphic cohomology in terms of homotopy groups of average cycles given in [25] (this description is a direct corollary of Theorem 2.5):

Corollary 2.9. *Let X be a smooth quasi-projective real variety. Then for any $k \geq \dim X - q + 1$*

$$L_q H\mathbb{R}_{q-k,q}(X) = \pi_k \mathcal{Z}_q(X_{\mathbb{C}})^{av}$$

In [25], we also prove the following equivariant generalization of Beilinson-Lichtenbaum conjecture, which allows us to compute Dos Santos's real Lawson homology groups (with $\mathbb{Z}/2$ coefficients) in some range of indexes in terms of Bredon homology groups:

Theorem 2.10. *Let X be a smooth quasi-projective real variety and $k > 0$. The cycle map*

$$\Phi : L^q H\mathbb{R}^{r,s}(X; \mathbb{Z}/2^k) \to H^{r,s}(X_{\mathbb{C}}(\mathbb{C}); \underline{\mathbb{Z}/2^k})$$

is an isomorphism if $r \leq 0$ (and $s \leq q$) and an injection if $r = 1$ (and $s \leq q$).

We also show that Dos Santos's real Lawson homology of a curve coincides, with integer coefficients, with Bredon homology groups:

Theorem 2.11. *Let X be a smooth real curve. Then*

$$L^q H\mathbb{R}^{r,s}(X; \mathbb{Z}) \to H^{r,s}(X_{\mathbb{C}}(\mathbb{C}); \underline{\mathbb{Z}})$$

is an isomorphism for any $q \geq 0$, $r \leq q$, and $s \leq q$.

3 Current and Future Work

3.1 Atiyah KR-theory

In a ongoing work with J. Heller [26] we use Theorem 2.7 to extend a theorem of Karoubi and Weibel. Although the Karoubi-Weibel is included in a general Quillen-Lichtenbaum conjecture, proved by Rosenshon and Ostavaer [43], the geometric proof in the lines given below reveals essential theorems about Dos Santos's real Lawson homology. In [30], Karoubi and Weibel proved that, in a range, there is an isomorphism between the algebraic K-theory of a smooth quasi-projective complex variety X with $\mathbb{Z}/2^k$ coefficients and the Atiyah K-theory with $\mathbb{Z}/2^k$ coefficients of the $\mathbb{Z}/2$-equivariant space of complex points of X. More specific they proved that the cycle map

$$K_n(X,\mathbb{Z}/2^k) \to KR^{-n}(X_{\mathbb{C}}(\mathbb{C}),\mathbb{Z}/2^k)$$

is an isomorphism for $n \geq dim(X)$ and a monomorphism for $n = dim(X)-1$, for any smooth quasi-projective variety X. As remarked in [30], the use of etale cohomology of a real variety hindered the authors to obtain the expected range for the isomorphism (see [43]), which is $dim(X)-1$. The main drawback of the above approach is that the etale cohomology of a real variety doesn't vanish in high degree cohomology. In the view of Theorem 2.7 we can replace the morphic cycle map into etale cohomology, with the cycle map into Bredon cohomology, and using this we can extend the Karoubi-Weibel theorem to the expected isomorphism range.

Moreover, the steps in our proposed geometric proof are the following results of independent interest. First we prove that there is a compatibility between the Poincare duality map between Bredon cohomology and Bredon homology and the Poincare duality map between equivariant morphic cohomology and real Lawson homology i.e.

Theorem 3.1. *Let X be a smooth d-dimensional real variety. The following square commutes, where the horizontal arrows are the Poincare duality isomorphisms*

$$\begin{tikzcd}
L^qH\mathbb{R}^{q-k,q-l}(X) = \pi_{k,l}\mathcal{Z}^q(X_{\mathbb{C}}) \arrow[r, "\mathcal{D}"] \arrow[d] & \pi_{k,l}\mathcal{Z}_{d-q}(X_{\mathbb{C}}) = L_{d-q}H\mathbb{R}_{d-q+k,d-q+l}(X) \arrow[d] \\
H^{q-k,q-l}(X(\mathbb{C}),\underline{\mathbb{Z}}) \arrow[r, "\mathcal{P}"] & H_{d-q+k,d-q+l}(X(\mathbb{C}),\underline{\mathbb{Z}}) \cong \pi_k\Omega^{d-q,d-q}\mathcal{Z}_0(X_{\mathbb{C}}).
\end{tikzcd}$$

This extends the ground-breaking theorems of Friedlander-Lawson [16] and Friedlander [18], about the similar compatibilities in the complex case for projective, quasi-projective varieties respectively.

Second, we prove that an iteration of the s-map operation defined by Dos Santos in [8] gives the Friedlander-Walker cycle map [23] i.e.

Theorem 3.2. *The Friedlander-Walker cycle map is compatible with the s-maps in the sense that coincides with the following composition*

$$L^qH^{r,s}(X) \xrightarrow{s} L^{q-1}H^{r,s}(X) \xrightarrow{s} ... \xrightarrow{s} H^{r,s}(X_{\mathbb{C}}(\mathbb{C}),\underline{\mathbb{Z}})$$

for any real quasi-projective variety X.

In the end we prove, using the approach of [14], that there is a spectral sequence between morphic cohomology of a real variety and semi-topological K-theory of a real variety and this spectral sequence is compatible with the spectral sequence of Dugger [10] between Bredon cohomology and Atiyah K-theory i.e.

Theorem 3.3. *For any smooth, projective real variety X and any abelian group A, there is a natural map of strongly convergent spectral sequences*

$$\begin{tikzcd}
E_2^{p,q}(sst) = L^{-q}H^{p,-q}(X,A) \arrow[r, Rightarrow] & K^{sst}_{-p-q}(X,A) \\
E_2^{p,q}(top) = H^{p,-q}(X_{\mathbb{C}}(\mathbb{C}),\underline{A}) \arrow[r, Rightarrow] & kr^{p+q}(X_{\mathbb{C}}(\mathbb{C}),\underline{A}).
\end{tikzcd} \tag{3.4}$$

inducing the usual maps on both E_2-terms and abutments.

The spectral sequence in Theorem 3.3 was also announced by E. Friedlander, C.Haesemeyer and M. Walker (unpublished).

3.2 S-Filtration for Reduced Lawson Homology

The s-map described in 1.2, gives a filtration on the Griffiths group of the complex variety X which is related to Nori's filtration [41] by the work of Friedlander [19]. It also gives a filtration on the singular homology of the topological space of complex points of X given by the images of compositions of s-maps. The latter filtration is known to be included in the Grothendieck's coniveau filtration [19]. A celebrated conjecture of Friedlander and Lawson says that this filtration is actually the same as the niveau filtration. With finite coefficients, this conjecture follows from the Beilinson-Lichtenbaum conjecture proved by Voevodsky and Rost (see for example [45]). With integer coefficients it follows from Suslin's conjecture.

Teh, using reduced Lawson homology, defined a filtration on the singular homology of the topological space of real points of an algebraic variety (in [47]). He showed that a similar s-operation, viewed as the multiplication with the first Stiefel-Whitney class of the canonical bundle, gives the following decompositions

$$RL_rH_n(X) \xrightarrow{s} RL_{r-1}H_n(X) \xrightarrow{s} .. \xrightarrow{s} RL_1H_r(X) \xrightarrow{s} H_r(X(\mathbb{R}),\mathbb{Z}/2). \tag{3.5}$$

This operation is interesting because it comes from the similarly constructed operation on real Lawson homology, which in turn gives the multiplication with -1 on etale cohomology of a real variety studied and used by Colliote-Thelene and Schneider [6] and Pedrini and Weibel [42]. We can see in 3.5 that taking the images of the compositions s^q we obtain a filtration on $H_r(X(\mathbb{R}),\mathbb{Z}/2)$, called s-filtration. In [47], Teh conjectured the following:

Conjecture 3.1. The s-filtration and niveau filtration on the singular homology of the topological space of real points of a smooth real variety coincide.

This conjecture suggests the existence of a Beilinson-Lichtembaum conjecture for real algebraic cycles. This is a natural continuation of Theorem 2.5 and we expect to be able to decide it using the tools constructed in [45].

3.3 Suslin's conjecture

In a work in progress [26], joint with J. Heller, we plan to prove that Suslin's conjecture is equivalent to a Bloch-Kato type of statement (see Conjecture 3.4) and then apply this result to prove that Suslin's conjecture is true for any smooth quasi-projective threefold. This will give the most general and supportive argument for Suslin's conjecture proved so far. It is expected that Suslin's conjecture, if true, would imply deep relations between families of algebraic cycles parametrized by spheres. For example, it is expected that the Weak Lefshetz theorem for singular cohomology has an extension to S^n families of algebraic cycles which is implied by (or possibly equivalent to) Suslin's conjecture.

Conjecture 1.2 is the most useful formulation for a case-by-case study of Suslin's conjecture. However, the equivalent formulation in terms of chain complexes is more suitable for a direct proof. This was introduced by Friedlander, Haesemeyer and Walker in [14]. Before stating it, let's introduce the notations.

Let $\mathbb{Z}(q)^{sst}$ denote the cochain complex of Zariski sheaves on $Sch_{\mathbb{C}}$,

$$\mathbb{Z}(q)^{sst}(-) = \mathrm{Hom}\left(- \times \Delta^{\bullet}_{top},\, C_0(\mathbb{P}^q)\right)^+ / \,\mathrm{Hom}\left(- \times \Delta^{\bullet}_{top},\, C_0(\mathbb{P}^{q-1})\right)^+ [-2q]. \tag{3.6}$$

Here $C_0(\mathbb{P}^n)$ is the Chow variety parameterizing effective zero-cycles on $\mathbb{P}^n$, $(-)^+$ is group completion, and the quotient is the quotient as Zariski sheaves of abelian groups. If X is smooth then $\mathbb{Z}(q)^{sst}(-)$ is pseudo-flasque and morphic cohomology may be computed as Zariski hypercohomology,

$$L^qH^p(X) = \mathbb{H}^p_{Zar}(X; \mathbb{Z}(q)^{sst}) = H^p(\mathbb{Z}(q)^{sst}(X)).$$

Write $S(n) = \tau_{\leq n}\mathbb{R}\varepsilon_*\mathbb{Z}$ and $\alpha_n : \mathbb{Z}(n)^{sst} \to S(n)$ for the above map. Because $\mathbb{R}\varepsilon_*\mathbb{Z}$ fulfill Zariski descent, we have

$$H^p_{sing}(X^{an}, \mathbb{Z}) = \mathbb{H}^p_{Zar}(X; \tau_{\leq n}\mathbb{R}\varepsilon_*\mathbb{Z}) = H^p(S(n)(X))$$

for any $p \leq n$. Suslin's Conjecture may be reformulated (see also Conjecture 1.2) as

Conjecture 3.2. (Suslin's Conjecture in weight n) The comparison map $\alpha_n : \mathbb{Z}^{sst}(n) \to S(n)$ is a quasi-isomorphism.

We can formulate the following conjecture, which we call morphic Bloch-Kato in weight n.

Conjecture 3.3. (MBK(n)) The comparison map $\alpha_n : \mathbb{Z}^{sst}(n) \to S(n)$ induces a surjection

$$\alpha_{m*} : \mathbb{H}^n_{Zar}(E, \mathbb{Z}(n)^{sst}) \to \mathbb{H}^n_{Zar}(E, \mathbb{Z}(n))$$

for any finite-type field extension $E/\mathbb{C}$.

We plan to prove the following conjecture:

Conjecture 3.4. If the morphic Bloch-Kato conjecture holds in all weights $q \leq n$, then Suslin's conjecture also holds in all weights $q \leq n$ as well.

We expect to be able to use this conjecture to prove Suslin's conjecture for any smooth projective threefold.

3.4 Quaternionic and Equivariant Morphic Cohomology

Another work in progress [12], joint with J. Heller and E.Friedlander, is a study of Beilinson-Lichtenbaum type of structure for quaternionic morphic cohomology and for equivariant morphic cohomology theories. One of the theorems that we plan to prove is:

Conjecture 3.5. Let X be a quaternionic smooth projective variety and $k > 0$. Then the map

$$\alpha_* : \pi_{r,s}(Z^n_{\mathbb{H}}(X) \otimes \mathbb{Z}/2^k) \to H^{2n+1-r,2n+1-s}_{Br}(X \times \mathbb{P}_{\mathbb{C}}(\mathbb{H}), \underline{\mathbb{Z}/2^k})$$

is an isomorphism for $r, s \geq 0$ and $r + s >> 0$.

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