Lecture 3

We will now briefly review the mathematical structure of QM, as well as the Dirac notation widely used in QIT. For this week, we will consider the case where our quantum systems are in completely known states, and well-isolated from any external environment — the pure state case. Next week, we will review how to include both noise and incomplete information in our description — the mixed state or density matrix case.

As we will see later, we can often map a noisy protocol onto an equivalent pure state protocol. This trick, called purification, can often simplify the mathematics, but it can also in some cases give extra insight: we explicitly include the external environment into which information is lost.

The mathematical structure of QM can be described by a set of four postulates. We will introduce these one at a time, along with the necessary notation.

Postulate 1: The <u>state</u> of any physical system is described by a d-dimensional complex vector in the Hilbert space $\mathbb{C}^d$.

For this introduction we will just look at the simplest case of d = 2: a quantum bit or <u>qubit</u>. Later we will see how to generalize straightforwardly to d > 2.

We write the state of a qubit as a column vec:

$$|\psi\rangle \equiv \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha|0\rangle + \beta|1\rangle$$

This is a <u>superposition</u> of $|0\rangle$ and $|1\rangle$

"ket vector"

label

probability amplitudes

standard orthonormal basis vectors:

$$|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

"the Z basis"

The probability amplitudes α and β are <u>complex</u> numbers. If $|\psi\rangle$ is <u>normalized</u> then

$$|\alpha|^2 + |\beta|^2 = 1$$

these are the probabilities of the outcomes for the standard basis (Z) measurement. Any choice of basis corresponds to some measurement.

For the qubit (but not a general d-vector) ③

there is a mapping from states $|\psi\rangle$ to points on the surface of a 3D Bloch sphere.

First note: global phases are physically irrelevant. So $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ represent the same physical state.

This means that we can choose to make α real.

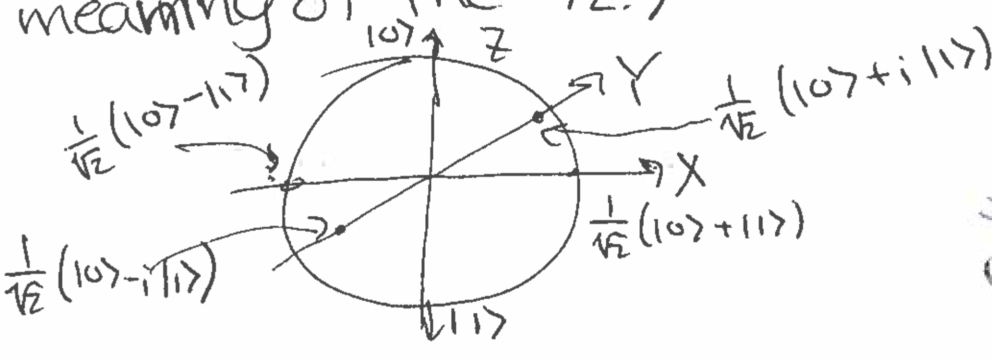
e.g., $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = r_0 e^{i\phi_0}|0\rangle + r_1 e^{i\phi_1}|1\rangle$ is equivalent to $e^{-i\phi_0}|\psi\rangle = r_0|0\rangle + r_1 e^{i(\phi_1-\phi_0)}|1\rangle$.

Second note: Normalization $|\alpha|^2 + |\beta|^2 = 1$ means we can parametrize α and β by two angles:

$$\alpha = \cos(\theta/2) \qquad 0 \le \theta \le \pi$$

$$\beta = \sin(\theta/2)\, e^{i\phi} \qquad 0 \le \phi \le 2\pi$$

The choice of $\theta/2$ here means that the points (θ, ϕ) uniquely cover the surface of a sphere. (There is also a physical meaning of the $1/2$.)



orthogonal states are 180° apart.

The corresponding row vectors (or 1-forms) are written

$$\langle\psi| = (|\psi\rangle)^\dagger = (\alpha^* \ \beta^*) \leftarrow \text{"bra vector"}$$

The inner product is written as a "bra" times a "ket" — a "bracket":

$$|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \quad |\phi\rangle = \begin{pmatrix}\gamma\\ \delta\end{pmatrix}$$

$$\langle\phi|\psi\rangle = (\gamma^* \ \delta^*)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \gamma^*\alpha + \delta^*\beta.$$

$$\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 = 1 \leftarrow \text{normalized!}$$

We can use the inner product to extract the prob. amplitudes in <u>any</u> orthonormal basis; e.g., $\langle 0|\psi\rangle = \alpha$, $\langle 1|\psi\rangle = \beta$.

There are ∞ly many bases. Some are used frequently, e.g.,

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) \leftarrow \text{"X basis"}$$

$$\langle\pm|\psi\rangle = \frac{\alpha \pm \beta}{\sqrt{2}}.$$

In addition to state vectors, there are linear transformations, or <u>operators</u>, written (or represented) as $d \times d$ matrices:

$$\text{e.g., } \hat{O} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \hat{O}^{+} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

Hermitian conjugate.

Matrices are often expressed in terms of <u>outer products</u>: $|\psi\rangle\langle\phi|$ is a $d \times d$ matrix.

$(|\psi\rangle\langle\phi|)^{+} = |\phi\rangle\langle\psi|$

In terms of a basis

$$\hat{O} = \sum_{i,j} o_{ij} |i\rangle\langle j| = a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1|$$

matrix elt.s

basis for $d \times d$ matrices

from example above.

If $|\psi\rangle$ is normalized, then $|\psi\rangle\langle\psi|$ is a <u>rank-one projector</u>:

$$(|\psi\rangle\langle\psi|)^2 = |\psi\rangle\underbrace{\langle\psi|\psi\rangle}_{1}\langle\psi| = |\psi\rangle\langle\psi|$$

$$(|\psi\rangle\langle\psi|)^{+} = |\psi\rangle\langle\psi| \leftarrow \text{Hermitian.}$$

Multiplying by a projector picks out a component in a particular subspace.

$$|0\rangle\underbrace{\langle 0|\psi\rangle}_{\alpha} = \alpha|0\rangle, \quad |1\rangle\underbrace{\langle 1|\psi\rangle}_{\beta} = \beta|1\rangle, \quad |\pm\rangle\langle\pm|\psi\rangle = \frac{\alpha \pm \beta}{\sqrt{2}}|\pm\rangle.$$

The usual matrix multiplication rules apply. Most matrices do not commute:

$$[A,B] \equiv AB - BA \leftarrow \text{"commutator"}$$
$$\{A,B\} \equiv AB + BA \leftarrow \text{"anticommutator"}$$

In the 2x2 case, a widely used set of matrices are I, X, Y, Z:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

X, Y and Z are the <u>Pauli matrices</u>. I, X, Y and Z are an orthogonal basis for the 2x2 matrices:

$$(A|B) \equiv \mathrm{Tr}\{A^{\dagger}B\} \leftarrow$$ The trace of a square matrix is the sum of the diagonal elements.

The Pauli matrices are <u>Hermitian</u>, <u>Unitary</u>, square to the identity, and have eigenvalues ± 1. They anticommute with each other, and have the algebraic properties

$$XY = iZ, \quad YZ = iX, \quad ZX = iY.$$

The matrices are also <u>traceless</u>: the trace is the sum of the eigenvalues, and $(+1)+(-1)=0$.

Postulate 2 Quantum states evolve linearly with time by unitary transformations:

$$|\psi(0)\rangle \longrightarrow |\psi(t)\rangle = U(t,0)\,|\psi(0)\rangle$$

Unitary matrices satisfy

$$U^\dagger U = U U^\dagger = I \iff U^\dagger = U^{-1}$$

Unitary matrices are normal, which means they have an orthonormal basis of eigenvectors (an eigenbasis). All such matrices can be diagonalized. The eigenvalues of U all have the form $\lambda_j = e^{i\theta_j} \iff |\lambda_j| = 1$.

$$U = \sum_j \underbrace{e^{i\theta_j}}_{\text{eigenval.}} \underbrace{|\phi_j\rangle\langle\phi_j|}_{\text{eigenvec.}}$$

The unitary evolution arises from the Schrödinger equation

$$\frac{d|\psi\rangle}{dt} = -\frac{i}{\hbar}\underbrace{H(t)}_{\text{Hamiltonian}}|\psi\rangle \implies |\psi(t)\rangle = U|\psi(0)\rangle$$

$$H = H^\dagger.$$

But we will not use this eqn in this class—only the unitaries that arise from it

Unitary transformations on a single qubit (2x2) are often called <u>quantum gates</u>. Note that the Paulis (X, Y, Z) are all quantum gates.

A commonly used gate is the <u>Hadamard</u>:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

H switches between the X and Z bases:

$$H|0\rangle = |+\rangle, \quad H|1\rangle = |-\rangle,$$
$$H|+\rangle = |0\rangle, \quad H|-\rangle = |1\rangle.$$

The most general 2x2 unitary takes the form of a <u>rotation</u> on the Bloch sphere:

$$\vec{n} = (n_x, n_y, n_z)$$

$$R_{\vec{n}}(\theta) = e^{i(\theta/2)\vec{n}\cdot\hat{\vec{\sigma}}}$$

$$\vec{n}\cdot\hat{\vec{\sigma}} = n_x X + n_y Y + n_z Z.$$

$$= \cos(\theta/2)\, I + i\sin(\theta/2)\, \vec{n}\cdot\hat{\vec{\sigma}}.$$

up to a global phase

this means that $(\vec{n}\cdot\hat{\vec{\sigma}})^2 = I$. $\rightarrow n_x^2 + n_y^2 + n_z^2 = 1$

Special cases are rotations around the X, Y, and Z axes

$$R_X(\theta) = e^{i(\theta/2)X}, \quad R_Y(\theta) = e^{i(\theta/2)Y}, \quad R_Z(\theta) = e^{i(\theta/2)Z}$$