Measurement and Interference

- Let us recall what we learned about projective measurements. Any ideal physical measurement on a system with a D-dimensional Hilbert space corresponds to an *observable*.
- An observable is a $D \times D$ Hermitian operator $\hat{O} = \hat{O}^\dagger$. Such operators are diagonalizable:

$$\hat{O} = \sum_j \lambda_j \hat{\mathcal{P}}_j, \quad \hat{\mathcal{P}}_j = \hat{\mathcal{P}}_j^\dagger, \quad \hat{\mathcal{P}}_j \hat{\mathcal{P}}_k = \hat{\mathcal{P}}_j \delta_{jk},$$

  with $M \leq D$ real distinct eigenvalues λ_j and M corresponding orthogonal projectors $\hat{\mathcal{P}}_j$.

- When a measurement is done, the *outcome* (i.e., the measured value) is one of the eigenvalues λ_j.
- After the measurement the system is left in an eigenstate of $\hat{O}$ corresponding to λ_j. This implies that if the measurement is immediately repeated, the same outcome will occur. The probability p_j of a particular outcome λ_j is determined by *Born's rule*.
- Let's see how this works for an observable that is *non-degenerate*, i.e., has $M = D$ distinct real eigenvalues λ_j.

Nondegenerate Observables

- In this case, each of the projectors $\hat{\mathcal{P}}_j = |\phi_j\rangle\langle\phi_j|$ is one-dimensional. The eigenstates $\{|\phi_j\rangle\}$ are determined uniquely (up to a phase) and are an orthonormal basis.
- We can write any state $|\psi\rangle$ in this basis:

$$|\psi\rangle = \sum_{j=1}^{D} \alpha_j |\phi_j\rangle,$$

  in which case the probability of outcome λ_j is $p_j = |\alpha_j|^2$ and the system is left in the eigenstate $|\phi_j\rangle$ after the measurement.
- Such a nondegenerate observable, with D distinct eigenvalues and one-dimensional spectral projectors, is called *complete*.

Degenerate Observables

- Suppose that the observable *is* degenerate, i.e., at least one of the projectors $\hat{\mathcal{P}}_j$ corresponding to an eigenvalue λ_j is more than one-dimensional, $\mathrm{Tr}\{\hat{\mathcal{P}}_j\} \equiv d_j > 1$.
- In this case, the basis of eigenvectors is not unique. What are the outcome probabilities? What state is the system left in?
- Let us choose an eigenbasis $|\phi_{jk}\rangle$, where states with the label j correspond to λ_j, and

$$\hat{\mathcal{P}}_j = \sum_k |\phi_{jk}\rangle\langle\phi_{jk}|.$$

  There are many ways of choosing this eigenbasis, in general, but it must be orthonormal.

- We can write the state in terms of this basis,

$$|\psi\rangle = \sum_{j=1}^{M} \sum_{k=1}^{d_j} \alpha_{jk} |\phi_{jk}\rangle.$$

The probability of outcome j is then $p_j = \sum_k |\alpha_{jk}|^2$, and the system is left in the state

$$|\psi_j\rangle = \sum_k (\alpha_{jk}/\sqrt{p_j}) |\phi_{jk}\rangle.$$

- The important thing to note is that this outcome does *not* depend on the particular choice of eigenbasis $|\phi_{jk}\rangle$. Choosing any orthonormal eigenbasis will give the same probabilities and the same final states.

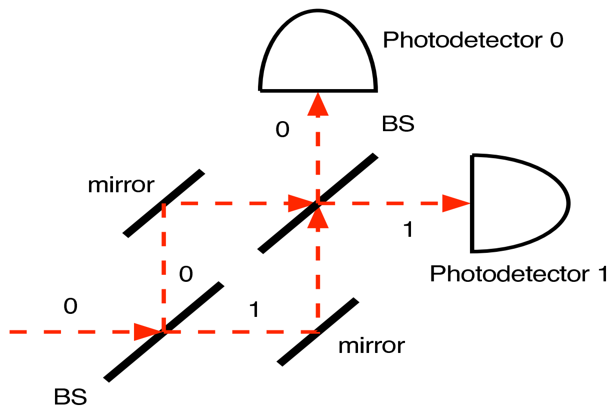
- We can see this by rewriting the expressions in terms of the projector $\hat{\mathcal{P}}_j$, which *is* uniquely determined:

$$p_j = \langle\psi|\hat{\mathcal{P}}_j|\psi\rangle, \quad \text{and} \quad |\psi_j\rangle = \hat{\mathcal{P}}_j|\psi\rangle/\sqrt{p_j}.$$

- Each eigenvalue λ_j corresponds to a *subspace* of Hilbert space; the probability of the outcome depends on the component of $|\psi\rangle$ in that subspace, and the system's state is orthogonally projected into the subspace.
- Note that these expressions include nondegenerate observables as a special case, where $\hat{\mathcal{P}}_j = |\phi_j\rangle\langle\phi_j|$.

Interference

Interference occurs when amplitudes combine to augment or suppress particular outcomes:



Suppose the system's initial state is $|0\rangle$. How does the state change as it passes through the two beam splitters?

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \rightarrow \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle + \frac{1}{2}|0\rangle - \frac{1}{2}|1\rangle = |0\rangle.$$

Interference between the two paths cancels the $|1\rangle$ terms while enhancing the $|0\rangle$ terms, so detector 0 always clicks.

Which-Way Information

What would happen if we did a projective measurement in between the two beam-splitters? That is, if we measured *which path* the photon took? This would give outcomes of 0 or 1 with equal probability.

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle,$$

$$\text{measurement} \rightarrow |0\rangle \qquad \qquad \text{measurement} \rightarrow |1\rangle$$

$$\text{OR}$$

$$|0\rangle \rightarrow \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \qquad \qquad |1\rangle \rightarrow \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle.$$

In both cases, there will be an equal probability of a click in detectors 0 or 1. *The measurement destroys interference*.

The Meaning of Superposition States

- One often sees superposition states like $(|0\rangle + |1\rangle)/\sqrt{2}$ said to mean that "the system is either in state 0 or state 1." But this interpretation cannot be correct! If it were, then measuring the system, to determine which state it really *was* in, should not change the outcome of *later* measurements. But it does! Measuring which arm of the interferometer held the photon lets detector 1 click when it never would without the measurement.
- This effect of measurement on interference is everywhere in QM. Since most QIP protocols rely on interference to work, including extra measurements can destroy their effectiveness. This includes interactions with the environment, which can effectively "measure" the system. This effect is called *decoherence*, and it is the enemy of quantum information!

Measurement and Entanglement

- How does measurement affect *entanglement*? Consider the maximally-entangled state $|\Phi_+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. If we measure the first system in the Z basis, we get results 0 and 1 with equal probability; after the measurement, the system is left in the product state $|00\rangle$ or $|11\rangle$. The same thing happens if we measure the second system in the Z basis.
- What if we measure the first system in the X basis? We can re-write this state

$$|\Phi_+\rangle = \frac{1}{2\sqrt{2}}\Big((|0\rangle+|1\rangle)\otimes(|0\rangle+|1\rangle)+(|0\rangle-|1\rangle)\otimes(|0\rangle-|1\rangle)\Big).$$

Once again, whatever the outcome, the final state will be a product.

- After any *complete local measurement* (i.e., a complete measurement of one of the subsystems) the system will be left unentangled. So we see that, as with interference, measurement can destroy entanglement.
- However, *joint* measurements of both subsystems can also *create* entanglement. For example, *Bell state measurement*. Even with an initial product state $|\psi\rangle \otimes |\phi\rangle$, the system is left in one of the Bell states:

$$|\Phi_+\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \qquad |\Phi_-\rangle = (|00\rangle - |11\rangle)/\sqrt{2}$$
$$|\Psi_+\rangle = (|01\rangle + |10\rangle)/\sqrt{2} \qquad |\Psi_-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}.$$

- In fact, *entanglement is not an observable*. There is no measurement which will reliably tell you if the pre-measurement state was entangled or not.

- We see both these effects at once in *quantum teleportation*. The initial state is $|\psi\rangle \otimes |\Psi_-\rangle$, where Alice has the first two q-bits and Bob has the third. There is no entanglement between Alice's first q-bit and the other two; but there *is* entanglement between her second q-bit and Bob's q-bit. Then Alice makes a Bell measurement on her two q-bits. Afterwards, there *is* entanglement between Alice's two q-bits, but no longer between Alice's q-bits and Bob's.
- The Bell state measurement in this case was a complete local measurement with respect to the division between Alice and Bob's systems, but an entangling joint measurement with respect to Alice's two subsystems.

Compatible measurements

- As stated before, if two observables $\hat{A}$ and $\hat{B}$ commute,

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} = 0,$$

  then $\hat{A}$ and $\hat{B}$ are compatible, and can be measured simultaneously. Let us see why.

- By the spectral theorem, both $\hat{A}$ and $\hat{B}$ can be decomposed in terms of projectors:

$$\hat{A} = \sum_{j=1}^{M_A} a_j \hat{\mathcal{P}}_j, \quad \hat{B} = \sum_{k=1}^{M_B} b_k \hat{\mathcal{P}}'_k.$$

- In order for $[\hat{A}, \hat{B}] = 0$ to be true, $[\hat{\mathcal{P}}_j, \hat{\mathcal{P}}'_k] = 0$ for all j, k.

- If two projectors $\hat{\mathcal{P}}_j$ and $\hat{\mathcal{P}}'_k$ commute, then their product is *also* a projector:

$$(\hat{\mathcal{P}}_j \hat{\mathcal{P}}'_k)^\dagger = \hat{\mathcal{P}}_j \hat{\mathcal{P}}'_k,$$

$$(\hat{\mathcal{P}}_j \hat{\mathcal{P}}'_k)^2 = (\hat{\mathcal{P}}_j)^2 (\hat{\mathcal{P}}'_k)^2 = \hat{\mathcal{P}}_j \hat{\mathcal{P}}'_k.$$

- This means that it is possible to write the spectral decompositions of $\hat{A}$ and $\hat{B}$ in terms of more *refined* projectors:

$$\hat{A} = \sum_{j=1}^{M_A} \sum_{k=1}^{M_B} a_j \hat{\mathcal{P}}_j \hat{\mathcal{P}}'_k, \quad \hat{B} = \sum_{j=1}^{M_A} \sum_{k=1}^{M_B} b_k \hat{\mathcal{P}}_j \hat{\mathcal{P}}'_k.$$

(Of course, many of the $\hat{\mathcal{P}}_j \hat{\mathcal{P}}'_k$ may actually be zero.)

- We have decomposed the two operators in terms of the *same* set of projectors. We can find an orthonormal basis of eigenvectors for *both* of these operators. Here's a simple example:

$$\hat{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

- If two observables $\hat{A}$ and $\hat{B}$ are *not* compatible (i.e., do not commute), they are *complementary*: they cannot be measured simultaneously. This means that while a state $|\psi\rangle$ can be written in terms of the eigenbasis of $\hat{A}$ *or* of $\hat{B}$, it cannot be written in terms of a simultaneous eigenbasis of both.

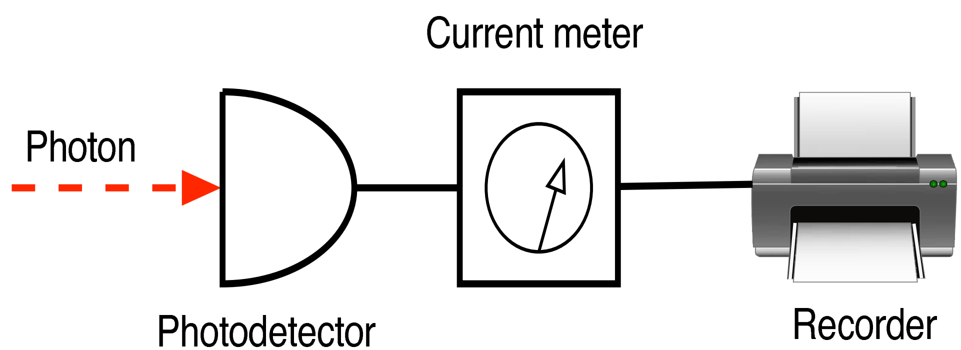
- Our prime example of complementary observables are the three Pauli operators $\hat{X}$, $\hat{Y}$ and $\hat{Z}$.
- Each of these is an observable; they have eigenbases $\{(|0\rangle \pm |1\rangle)/\sqrt{2}\}$, $\{(|0\rangle \pm i|1\rangle)/\sqrt{2}\}$, $\{|0\rangle, |1\rangle\}$. If a spin is in one of these eigenstates, it has a definite value of the corresponding observable; but the other two are *completely undetermined*: any measurement of them has a $50/50$ chance of seing either possible result.
- Complementarity is what makes BB84 work. Alice chooses definite values for one of two complementary variables to transmit her bits. Eve cannot measure one variable without disturbing the other.

Experimental measurements

- The mathematical formulation of projective measurement is extremely elegant. In this idealized description, measurements disturb the system as little as possible; they are repeatable, and always yield definite (though not deterministic) outcomes.
- In a commonly used phrase, ideal projective measurements are *quantum nondemolition* (QND) measurements.
- In the real world, things are not so nice.

Photodetection

- A good example of this is the *photodetector*. While there are various kinds of photodetectors, they for the most part work in roughly the same way. The photon is absorbed by the detector; the energy liberates some electric charges, which produce a measurable current (called the *photocurrent*) that is detected by a meter.

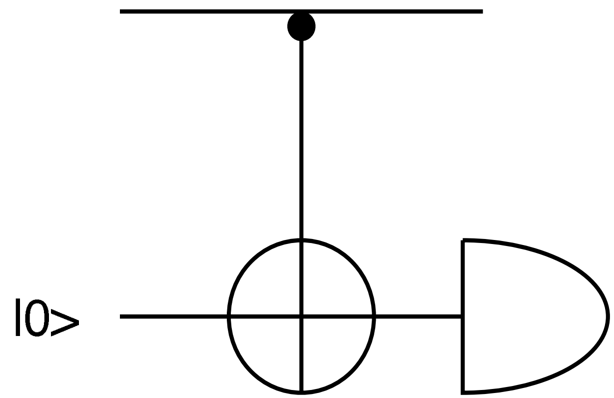


- Unfortunately, this is very far from being a QND measurement! It destroys the photon that is measured.

- Detecting, e.g., which arm of a Mach-Zender interferometer a photon is in *without* absorbing it is extremely difficult. Often the best that can be done is determining that the photon *isn't* in one arm, by the failure of a photodetector to click.
- The spin-1/2 can be measured by a Stern-Gerlach device. Depending on the orientation of the spin, the atom it is attached to will be deflected up or down. However, we don't know the outcome until the atom strikes a photographic plate, at which point it is lost.
- In some cases, we don't need ideal measurements. Many QIP protocols only do measurements at the end. In principle, it is *always* possible to defer all measurements to the end of the protocol. This is the *principle of deferred measurement*. Let's see how it works.

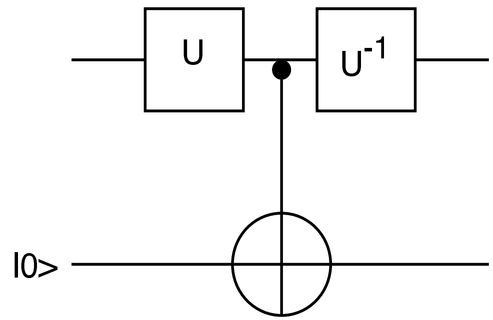
Deferred Measurement

Suppose we wish to measure a q-bit in the Z basis, but we still need to use it after the measurement. How can we get around this problem? Consider this quantum circuit:

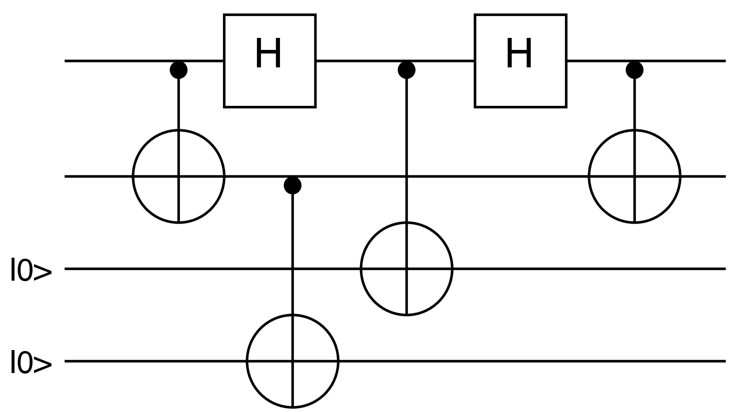


If the first bit is in the state $|0\rangle$ the second bit will be left in state $|0\rangle$; if it is in state $|1\rangle$, the second bit will be flipped. We could then measure the second bit, and we will have, indirectly, measured the first—without destroying it. Moreover, we can wait to measure the second bit until the procedure is over without altering the results.

What if we want to measure in other than the standard Z basis? Let $\hat{U}$ change from the basis we wish to measure to the Z basis, and carry out this circuit:



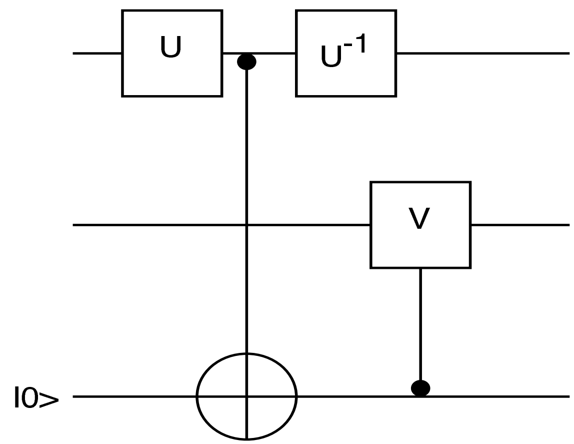
The same trick works for measurements on more than one quantum bit. Here is a circuit for a Bell state measurement:



Conditioning Actions

- In some protocols, we may use the outcome of a measurement to determine an action to perform. For instance, in the quantum teleportation protocol, the outcome of Alice's Bell state measurement determines which unitary transformation Bob must perform.
- Even in this case, it is often possible to defer all measurements to the end of the procedure. We can replace this conditioned operation by a *controlled* operation.

- Suppose we wish to measure a q-bit in some basis, and if the result is a 1, do a unitary transformation $\hat{V}$ on another q-bit.



- The deferred measurement principle uses the fact that we can do measurements *indirectly*.

Indirect Measurements

Such indirect measurements take the following form:

1. Prepare an *extra* system (e.g., another q-bit) in a *known* initial state (e.g., $|0\rangle$). This extra system is often referred to as an *ancillary system* or *ancilla*.
2. Have the system and the ancilla interact by carrying out some circuit.
3. Measure the ancilla. This will give information about the system, thereby indirectly measuring it as well.

In practice, virtually all real measurements are indirect. Even our own eyesight works indirectly, by intercepting light which has bounced off of the object we are viewing.

Example of Indirect Measurement

Here is an example of an indirect measurement used in quantum information:

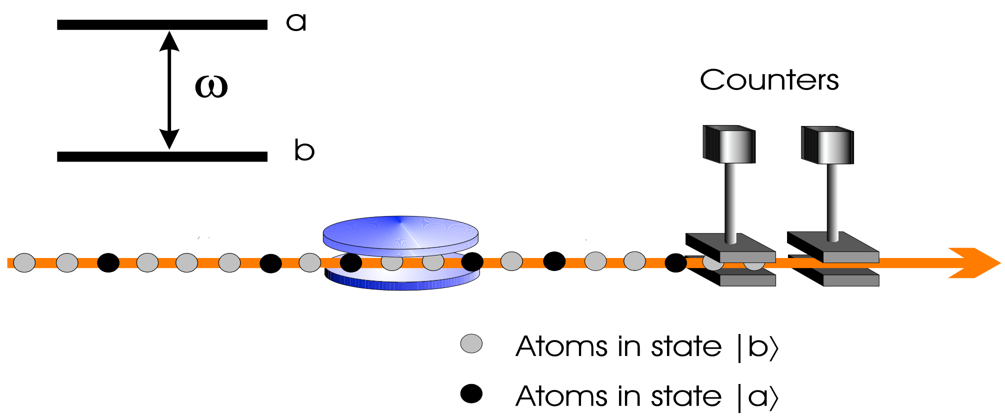


Figure 1 - Kist - PRA

The system we wish to measure is the microwave mode in the superconducting cavity. We probe it by preparing atoms in a particular state and sending them through the cavity; while inside, they interact with the microwaves, and are measured when they emerge.

Generalized Measurements

- In fact, the use of ancillas gives us more than just a way of doing projective measurements. It enables us to carry out a broad range of actions, of which projective measurements are just a special case. These more general procedures are called *generalized measurements* (or sometimes *quantum operations*).
- *Next time: Generalized measurements.*