Supplement to Stable Farsighted Coalitions in Competitive Markets - Technical Appendix

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Proposition 1

1. Suppose that \( i \in Z_k, j \in Z_m \), and \( |Z_k| < |Z_m| \). Then, \( \Pi^Z_i > \Pi^Z_j \).

2. Suppose that \( j \in Z_k \) leaves the coalition, changing the coalition structure to \( Z' = \{Z_1, \ldots, Z_k \setminus \{j\}, \{j\}, \ldots, Z_l\} \). Then, the profit for members of the coalition \( Z_k \setminus \{j\} \) decreases with respect to the profit realized in the coalition \( Z_k \).

3. Let \( Z_1 = \{Z_1, \ldots, Z_m\} \) and \( Z_2 = \{Z_1 \cup Z_2, Z_3, \ldots, Z_m\} \) be the status quo position and the new position in which two of the coalitions merge, respectively. Then \( \forall n, \alpha > 0 \), we have \( \Pi^Z_{Z_1 \cup Z_2} \geq \Pi^Z_{Z_1} + \Pi^Z_{Z_2} \). Further, if \( |Z_1| \geq |Z_2| \) then \( \forall \alpha \geq 0, \Pi^Z_{Z_1} \geq \Pi^Z_{Z_2} \) and \( \exists \alpha^* \) such that \( \forall \alpha \leq \alpha^* \), \( \Pi^Z_{Z_1} \geq \Pi^Z_{Z_2} \).

To prove the first item in Proposition 1, we need the following very useful lemma, which states that smaller coalitions charge lower prices in equilibrium.

Lemma 1 Suppose that \( i \in Z_k, j \in Z_m \), and \( |Z_k| < |Z_m| \). Then, \( p_i^* < p_j^* \).

Proof of Lemma 1: For each member, \( i \), of a coalition \( Z_k \) in partition \( Z \),

\[
\Pi^Z_i = p_i D^Z_i = p_i \left[ A - (1 + \alpha)p_i + \frac{\alpha |Z_k|}{n} p_i + \frac{\alpha}{n} \sum_{j \notin Z_k} p_j \right]
\]

where \( D^Z_i \) denotes demand for a member, \( i \), of coalition \( Z_k \) in partition \( Z \). The first order condition for a coalition member can be written as \( D^Z_i + p_i \left[ -(1 + \alpha) + \alpha |Z_k|/n \right] = 0 \), hence in optimality

\[
D^Z_i = p_i^* \left[ (1 + \alpha) - \alpha \frac{|Z_k|}{n} \right]. \tag{A1}
\]
Suppose that $p_i^* \geq p_j^*$ and $i \in Z_k, j \in Z_m$, and $|Z_k| < |Z_m|$. Then, since it follows from (1) that $D_i(p_1, \ldots, p_n) - D_j(p_1, \ldots, p_n) = (p_j - p_i)(1 + \alpha)$, $j$ faces larger demand than $i$, $D_j \geq D_i$. On the other hand, because $|Z_k| < |Z_m|$ and $p_i^* \geq p_j^*$, it follows from (A1) that $D_i > D_j$, which is a contradiction. Therefore, $p_i^* < p_j^*$.

**Proof of Proposition 1:**

1. We show in Lemma 1 that $p_i^* < p_j^*$ when $i \in Z_k, j \in Z_m$, and $|Z_k| < |Z_m|$. Therefore,

$$\Pi_j(p_i^*, p_j^*, p_{-i,j}) < \Pi_j(p_j^*, p_j^*, p_{-i,j}) = \Pi_i(p_i^*, p_j^*, p_{-i,j}) < \Pi_i(p_i^*, p_j^*, p_{-i,j}),$$

where the first inequality follows from $p_i^* < p_j^*$, and the second one from the definition of the NE.

2. Let us denote by $Z$ the original coalition structure, by $Z'$ the structure obtained when $j \in Z_k$ leaves the coalition, and by $s \in Z_k \setminus \{j\}$ an arbitrary coalition player. According to Proposition 1, $j$'s profit in $Z'$ is larger than the profit of the remaining coalition members,

$$\Pi_{Z'}^s \leq \Pi_{Z}^j.$$

When all $|Z_k|$ members in $Z$ select a price to maximize their profits, the total profit realized by all players is higher than the one realized when $|Z_k| - 1$ members select their price independently of the player $j$ in $Z'$,

$$\sum_{i \in Z_k} \Pi_i^Z = |Z_k| \cdot \Pi_{Z}^s \geq \sum_{i \in Z_k} \Pi_i^{Z'} = (|Z_k| - 1)\Pi_{Z}^{Z'} + \Pi_j^{Z'}.$$

Now, the above two inequalities together imply $\Pi_{Z}^{Z'} \leq \Pi_{Z}^s$.

3. Suppose that a member of coalition $Z_i \in Z_1$ charges price $p_i$, and a member of $Z_1 \cup Z_2 \in Z_2$ charges $p^*$. The first order condition before and after merger can be written as

$$\frac{d\Pi_{Z_1}^Z}{dp_i} = D_{Z_1}^Z + p_i \left[ \frac{\alpha |Z_i|}{n} - (1 + \alpha) \right] = 0 \quad (A2)$$

$$\frac{d\Pi_{Z_1 \cup Z_2}^Z}{dp^*} = D_{Z_1 \cup Z_2}^Z + p^* \left[ \frac{\alpha |Z_1 \cup Z_2|}{n} - (1 + \alpha) \right] = 0. \quad (A3)$$

Since $|Z_1| > |Z_2|$, we must have $p_1 > p_2$. If we assume $p^* = p_1$, then (A3) is equivalent to

$$D_{Z_1}^Z + (p_1 - p_2)|Z_2| \frac{\alpha}{n} - p_1 \alpha + p_1 \alpha \frac{|Z_1 \cup Z_2|}{n} \geq 0.$$
It follows from concavity of $\Pi^Z_{Z_1 \cup Z_2}$ that $p_1 < p^*$, which implies the result. By continuity, we can show existence of $\alpha^*$.

Proposition 2

1. The prices set by both coalition members and noncoalition members are higher with respect to the case where all players act independently (i.e., no alliances are formed).

2. Equilibrium prices for both coalition members and noncoalition members increase with the size of the coalition.

3. When $k > 1$, noncoalition members generate larger profit than retailers in a coalition.

4. Profit for coalition members increases with the size of the coalition.

Proof of Proposition 2:

1. It follows from (2), (3), and (4):

$$p^*_C - p^*_0 = \frac{A\alpha(k - 1)[2n + \alpha(n + k - 1)]}{D(k)} > 0, \quad p^*_C - p^*_\bar{C} = \frac{A\alpha^2k(k-1)}{D(k)} > 0.$$

2. First note that

$$\frac{\partial D(k)}{\partial k} = -2\alpha - 2\alpha^2 + 2(2 - k)n \leq 0.$$ 

Thus, $D(k)$ is decreasing in $k$, hence $p^*_C$ is increasing. In addition,

$$\frac{\partial p^*_C}{\partial k} = A \frac{2\alpha^2(k-1) + \alpha^3(-2 + 4k - \frac{k^2}{n})}{D(k)^2} \geq 0.$$

3. Suppose player $i$ belongs to the coalition $C$, and player $j$ does not. Observe that profits for retailer $i$ and $j$, $\Pi_i(p_1, \ldots, p_n)$ and $\Pi_j(p_1, \ldots, p_n)$, are structurally similar except for $p_i$ and $p_j$. Denote by $p_{-\{i,j\}} = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n)$. Then,

$$\Pi_i(p^*_i, p^*_j, p^*_{-\{i,j\}}) < \Pi_i(p^*_i, p^*_i, p^*_{-\{i,j\}}) = \Pi_j(p^*_i, p^*_j, p^*_{-\{i,j\}}) < \Pi_j(p^*_i, p^*_j, p^*_{-\{i,j\}}),$$

where the first inequality follows from $p^*_C > p^*_\bar{C}$, and the second one from the definition of the NE.
4. It is easy to evaluate that, for any coalition member $i$,
\[
\frac{\partial \Pi_i}{\partial k} \bigg|_{(p_1^*,\ldots,p_n^*)} = \alpha^2 A p_C^* \left[ 4n^2(k-1) + 2\alpha n(5nk - 4n - 3k^2 + k + 1) \right. \\
\left. + \alpha^2(n-k)(6nk - 4n - 3k + 2) \right].
\] (A4)

Let us denote
\[
G(k) = 4n^2(k - 1) + 2\alpha n(5nk - 4n - 3k^2 + k + 1) + \alpha^2(n-k)(6nk - 4n - 3k + 2).
\]
Then,
\[
G'(k) = 4n^2 + 2\alpha n(5n - 6k + 1) + \alpha^2(6n^2 - 12nk + n + 6k - 2),
\]
and
\[
G''(k) = -6\alpha(2n + 2\alpha - \alpha) \leq 0,
\]
hence $G(k)$ is concave in $k$. Furthermore, $G(1) = 2\alpha n(n-1) + \alpha^2(n-1)(2n-1) \geq 0$, and $G(n) = 4n^2(n-1) + 2\alpha n(2n^2 - 3n + 1) \geq 0$, hence $G(k) \geq 0$ for any $k = 1, \ldots, n$. Thus, the RHS in (A4) is nonnegative, and $i$’s profit increases in $k$.

**Proposition 3** For any $n \geq 4$, there is an $\alpha(n)$, defined by (5), such that for $\alpha \leq \alpha(n)$, a player realizes higher profit by staying independent and not joining the grand coalition.

**Proof of Proposition 3:** It is easy to evaluate that, for a noncoalition member,
\[
p_C^{\alpha(n-1)} = A \frac{\alpha(n+1) + 2n}{3\alpha^2(1 - \frac{1}{n}) + 4n\alpha + 4n}, \\
D_C^{\alpha(n-1)} = A \frac{n(\alpha + 2)(\alpha + 1) - \alpha - \frac{\alpha^2}{n}}{3\alpha^2(1 - \frac{1}{n}) + 4n\alpha + 4n},
\]
while for a coalition member,
\[
p_C^{\alpha(n)} = A \frac{\alpha(2n-1) + 2n}{3\alpha^2(1 - \frac{1}{n}) + 4n\alpha + 4n}, \\
D_C^{\alpha(n)} = A \frac{2n(\alpha + 1) + \alpha(2\alpha + 1) - \frac{\alpha^2}{n}}{3\alpha^2(1 - \frac{1}{n}) + 4n\alpha + 4n}.
\]

Therefore, the difference between his profits in these coalition structures can be written as
\[
\Pi_C^{\alpha(n-1)} - \Pi_C^{\alpha(n)} = A^2 \frac{(n-1)[-9(n-1)\alpha^2 + 4n(n^2 - 4n + 1)\alpha + 4n^2(n-3)]}{4(3\alpha^2(1 - \frac{1}{n}) + 4n\alpha + 4n)^2}.
\]
To determine when this difference is positive, let us first define a quadratic function $G(\alpha) = -9(n-1)\alpha^2 + 4n(n^2 - 4n + 1)\alpha + 4n^2(n-3)$. This is a concave function that satisfies $G(0) > 0$. Thus, after determining its zeros, we can conclude that $G(\alpha) \geq 0$ for
\[
\alpha \leq \frac{2n(n^2 - 4n + 1) + 2n\sqrt{(n^2 - 4n + 1)^2 + 9(n^2 - 4n + 3)}}{9(n-1)} = \alpha(n),
\] (A5)

or, in other words, $\Pi_C^{\alpha(n-1)} \geq \Pi_C^{\alpha(n)}$ when (A5) holds.
Proposition 4 For $n$ large, every $Y \not\in$ the LCS is indirectly dominated by either the grand coalition or a basic coalition structure in the substitute model.

Proof of Proposition 4: Recall that $Z$ denotes the set of all possible coalition structures. For any $U \subseteq Z$, define a relation $\sim_U \subseteq Z \times Z$ as follows:

$$(X, Y) \in \sim_U \text{ if } Y \in U \Rightarrow X \ll Y, \text{ and } Z \in U \text{ and } Y \ll Z \Rightarrow X \ll Z.$$ 

Let $X \subseteq Z$, $X = \{\text{grand coalition + set of all basic coalitions}\}$. Define $\triangleleft_X \subseteq Z \times X$ as

$$(X, Y) \in \triangleleft_X \text{ if } Y \in X \Rightarrow X \ll Y \text{ and } Z \in X \text{ and } Y \ll Z \Rightarrow X \ll Z.$$ 

Define $\Diamond_U = \sim_U \cap \triangleleft_X \subseteq Z \times Z$. We can conclude that $\Diamond_U$ is finite and transitive. Given any $Y \subseteq Z$, let $M(Y, \Diamond Y) = \{A \in Y : B \in Y \text{ such that } (A, B) \in \Diamond Y\}$. We can show that $M$ is nonempty. Define $\ell : 2^Z \to 2^Z$ as $\ell(A) = M(A, \Diamond A)$. We then have that there exists $W$ such that $\ell(W) = W \neq \emptyset$. This is true because Proposition 1 implies $\ell^{i+1}(A) \subset \ell^i(A) \forall A, \forall$ large $n$, and we are guaranteed $W$’s existence because $Z$ is finite.

Now we can show that if $X \in Z \setminus W$, there is $Y \in W$ such that $(X, Y) \in \sim_W$ and that $W \subset M(Z, \Diamond W)$. Using Tarski (1955) we can show that if $f : 2^Z \to 2^Z$ is defined as $f(X) = \{Z \in Z \text{ such that } \forall V, S, \text{ such that } Z \rightarrow_S V, \exists B \in X, \text{ where } V = B \text{ or } V \ll B, \text{ and } Z \not\ll_S B\}$. For large $n$ there is an $\alpha$ such that $V \subseteq f(V)$ and thus $V \subseteq LCS$.

Proposition 5 When $n = 3$, the only stable coalition structure is the alliance of all players, $Z_3^3$.

Proof of Proposition 5: It follows from Proposition 2 that $\Pi^{Z_3^3} < \Pi^{Z_3^2} < \Pi^{Z_3^2}$, and $\Pi^{Z_3^2} < \Pi^{Z_3^3}$.

We need to find the relationship between the profit an independent player makes in $Z_2^3$ and the profit he can make in the grand coalition. When a player is independent, while the other two players form a coalition, his price at optimality and the corresponding demand are given by

$$p^{Z_3^3}_C = A \frac{6 + 4\alpha}{12 + 12\alpha + 2\alpha^2}, \quad D^{Z_3^3}_C = A \frac{6 + 8\alpha + \frac{8}{3}\alpha^2}{12 + 12\alpha + 2\alpha^2}.$$ 

Therefore,

$$\Pi^{Z_3^3} - \Pi^{Z_3^2} = A^2 \frac{\alpha^4 + \frac{4}{3}\alpha^3}{(12 + 12\alpha + 2\alpha^2)^2} \geq 0,$$

hence the profit in the grand coalition dominates the profit he can make in any other coalition structure,

$$\Pi^{Z_3^3} < \Pi^{Z_3^2} < \Pi^{Z_3^2} < \Pi^{Z_3^3}.$$
or if we write it in terms of players’ preferences,
\[ Z^3_1 \prec_i Z^3_2 \prec_i Z^3_3 \quad \forall i \in N. \quad (A6) \]

1. First, we show that the grand coalition belongs to the LCS. Consider a deviation by an arbitrary coalition \( S \subset N \), where \( S \) can consist of either 1 or 2 players, which leads to a coalition structure \( Z^3_S \). Any such deviation can be followed by another deviation of all three players, \( Z^3_3 \rightarrow S Z^3_S \rightarrow_{1,2,3} Z^3_3 \). Clearly, it follows from \((A6)\) that \( Z^3_S \prec_S Z^3_3 \) for any \( S \subset N \), while at the same time, \( Z^3_3 \nprec_S Z^3_3 \). In other words, if \( Z = Z^3_3 \), \( V = Z^3_S \), and \( B = Z^3_3 \), then \( Z^3_S = V \nprec B = Z^3_3 \), and \( Z^3_3 = Z \nprec B = Z^3_3 \). Thus, any possible deviation from the grand coalition is deterred.

2. Now, assume that the current status quo is \( Z^3_3 \), and consider a deviation by all three players, in which they form the grand coalition, \( Z^3_3 \rightarrow_{1,2,3} Z^3_3 \). It follows from \((A6)\) that \( Z^3_S \prec_S Z^3_3 \) for any \( S \subset N \), hence we cannot find a coalition structure that can be obtained by a deviation from \( Z^3_3 \) which strictly dominates \( Z^3_3 \) for any subset of players. That is, if \( Z = Z^3_1 \), \( V = Z^3_3 \) and \( S = \{1,2,3\} \), then clearly \( Z \nrightarrow S V \), but we cannot find a coalition structure \( B \) such that \( Z^3_3 = V \nprec B = Z^3_3 \). In addition, \((A6)\) implies that \( Z^3_1 \prec_{\{1,2,3\}} Z^3_3 \), hence if \( V = B \), then \( Z \prec_S B \).

Therefore, when \( Z = Z^3_1 \), \( V = Z^3_3 \) and \( S = \{1,2,3\} \), we cannot find a coalition structure \( B \), where \( V = B \) or \( V \nprec B \), such that \( Z \nprec S B \), hence \( Z^3_1 \) cannot be stable. Similar analysis can be used to show that \( Z^3_2 \) cannot be stable.

**Proposition 6** When \( n = 4 \), the coalition structure with no alliances, \( Z^4_1 \), and a coalition structure with an alliance of two players, \( Z^4_2 \), are never stable.

1. If \( \alpha > 1.864 \), the only stable coalition structure is the alliance of all players, \( Z^4_4 \).

2. If \( 1.2516 < \alpha \leq 1.864 \), the coalition structure \( Z^4_3 \), where an alliance of three players is formed, and the grand coalition, \( Z^4_4 \), are stable.

3. If \( \alpha \leq 1.2516 \), the coalition structure \( Z^4_2,2 \), where two alliances of two players are formed, the coalition structure \( Z^4_3 \), and the grand coalition \( Z^4_4 \) are all stable.

**Proof of Proposition 6:** It follows from Proposition 2 that
\[ \Pi^{Z^4_3} < \Pi^{Z^4_2}_C < \Pi^{Z^4_2}_C, \quad \Pi^{Z^4_3}_C < \Pi^{Z^4_3}_C, \quad \text{and} \quad \Pi^{Z^4_3}_C < \Pi^{Z^4_2}_C. \]
We need to find the relationship between $\Pi^Z_C$ and $\Pi^Z_4$, the relationship between $\Pi^Z_2$ and $\Pi^Z_4$, and the relationship between the profit realized in $Z_{2,2}$ and that realized in other coalition structures.

The optimal prices charged by the players when a coalition of three retailers is formed and demands that they face are given by

$$
p^Z_3 = A \frac{8 + 7\alpha}{16 + 16\alpha + \frac{9}{4}\alpha^2}, \quad p^Z_3 = A \frac{8 + 5\alpha}{16 + 16\alpha + \frac{9}{4}\alpha^2},$$
$$
D^Z_3 = A \frac{8 + 9\alpha + \frac{3}{4}\alpha^2}{16 + 16\alpha + \frac{9}{4}\alpha^2}, \quad D^Z_3 = A \frac{8 + 11\alpha + \frac{15}{4}\alpha^2}{16 + 16\alpha + \frac{9}{4}\alpha^2}.
$$

It is, then, easy to verify that

$$
\Pi^Z_4 - \Pi^Z_3 = A^2 \frac{\alpha^2(81\alpha^2 - 3\alpha - 12)}{4(16 + 16\alpha + \frac{9}{4}\alpha^2)^2},
$$

which is positive for $\alpha \geq 1.864$. Next, for one two-player coalition, the optimal prices charged in optimality and corresponding demands are given by

$$
p^Z_2 = A \frac{8 + 7\alpha}{16 + 18\alpha + 4\alpha^2}, \quad p^Z_2 = A \frac{8 + 6\alpha}{16 + 18\alpha + 4\alpha^2},$$
$$
D^Z_2 = A \frac{8 + 11\alpha + \frac{7}{2}\alpha^2}{16 + 18\alpha + 4\alpha^2}, \quad D^Z_2 = A \frac{8 + 12\alpha + \frac{9}{2}\alpha^2}{16 + 18\alpha + 4\alpha^2}.
$$

Therefore,

$$
\Pi^Z_4 - \Pi^Z_2 = A^2 \frac{\alpha^3(59\cdot\frac{5}{16}\alpha^4 + 505\cdot\frac{1}{4}\alpha^3 + 1312\alpha^2 + 1376\alpha + 512)}{(16 + 16\alpha + \frac{9}{4}\alpha^2)^2(16 + 18\alpha + 4\alpha^2)^2} \geq 0,
$$

hence

$$
\Pi^Z_2 \leq \Pi^Z_4.
$$

Finally, for two two-player coalitions,

$$
p^Z_{2,2} = A \frac{2}{4 + \alpha}, \quad D^Z_{2,2} = A \frac{2 + \alpha}{4 + \alpha},
$$

hence

$$
\Pi^Z_4 - \Pi^Z_{2,2} = A^2 \frac{\alpha^2(81\alpha^2 - 3\alpha - 12)}{4(\alpha + 4)^2} \geq 0,
$$

$$
\Pi^Z_2 - \Pi^Z_{2,2} = A^2 \frac{\alpha^2(81\alpha^2 + 16\alpha + 112)}{(\alpha + 4)^2(16 + 16\alpha + \frac{9}{4}\alpha^2)^2} \geq 0,
$$

$$
\Pi^Z_4 - \Pi^Z_{2,2} = A^2 \frac{\alpha^2(17\alpha^3 + 43\alpha^2 - 4\alpha - 16)}{(\alpha + 4)^2(16 + 16\alpha + \frac{9}{4}\alpha^2)^2} \geq 0 \text{ for } \alpha \geq 1.2516,
$$

$$
\Pi^Z_{2,2} - \Pi^Z_4 = A^2 \frac{\alpha^2(5\alpha^3 + 28\alpha^2 + 40\alpha + 16)}{(\alpha + 4)^2(16 + 18\alpha + 4\alpha^2)^2} \geq 0.
$$
Therefore, for $\alpha \leq 1.2516$,
\[
\Pi_{Z_4^1} < \Pi_{Z_4^2} < \Pi_{Z_4^3} < \Pi_{Z_4^{1,2}} < \Pi_{Z_4^4} < \Pi_{Z_4^5},
\]
and
\[
Z_1^4 \prec_i Z_2^4 \prec_i Z_3^4, \quad Z_1^4 \prec_i Z_2^4 \prec_i Z_3^4 \prec_i Z_4^4, \quad Z_3^4 \prec C Z_2^{1,2} \prec_i Z_4^4 \prec C Z_3^4 \forall i \in N, \quad (A7)
\]
for $1.2516 < \alpha \leq 1.864$,
\[
\Pi_{Z_4^1} < \Pi_{Z_4^2} < \Pi_{Z_4^3} < \Pi_{Z_4^{1,2}} < \Pi_{Z_4^4} < \Pi_{Z_4^5},
\]
and
\[
Z_1^4 \prec_i Z_2^4 \prec_i Z_3^4 \prec_i Z_4^4, \quad Z_1^4 \prec_i Z_2^4 \prec_i Z_3^4 \prec_i Z_4^4 \forall i \in N, \quad Z_3^4 \prec C Z_4^4 \prec C Z_4^4, \quad (A8)
\]
and for $\alpha > 1.864$,
\[
\Pi_{Z_4^1} < \Pi_{Z_4^2} < \Pi_{Z_4^3} < \Pi_{Z_4^{1,2}} < \Pi_{Z_4^4} < \Pi_{Z_4^5},
\]
and
\[
Z_1^4 \prec_i Z_2^4 \prec_i Z_3^4 \prec_i Z_4^4 \forall i \in N. \quad (A9)
\]

1. Let us first suppose $\alpha > 1.864$. Then, it follows from (A9) that each player realizes the highest profit in the grand coalition. Therefore, an argument similar to the one used in the proof of Proposition 5 may be used to show that the grand coalition is the only stable coalition structure.

2. Next, assume that $1.2516 < \alpha \leq 1.864$.

(a) As before, assume that the status quo is the grand coalition, $Z_4^4$. Since (A8) implies $Z_1^4 \prec_i Z_2^4 \prec_i Z_2^4 \prec_i Z_4^4$ for all $i$, for any coalition structure $\mathcal{V} \neq Z_4^4$, obtained when some coalition $S$ deviates from $Z_4^4$, letting $B = S = Z_4^4$ satisfies $\mathcal{V} \ll B = Z_4^4$ and $Z = Z_4^4 \not\prec S B = Z_4^4$. Since $Z_3^4 \prec C Z_4^4$, a similar argument can be used when any three players deviate from the grand coalition to form a three-player coalition. If a single player, say 4, decides to deviate, that is, $S = \{4\}$ and $\mathcal{V} = Z_3^4$, consider a following sequence of deviations:
\[
Z_4^4 \rightarrow_4 \{(123), 4\} = Z_3^4 \rightarrow_3 \{(12), 3, 4\} = Z_2^4 \rightarrow_{1,2,3,4} Z_4^4. \quad (A10)
\]
Then, $B = Z = Z_4^1$, $Z = Z_4^1 \not\ll_S B = Z_4^1$, while $Z_3^4 = V \ll B = Z_4^1$ because of (A10) and because $Z_3^4 \prec_3 Z_4^1$ and $Z_2^4 \prec_{\{1,2,3,4\}} Z_4^1$, which follows from (A8). Thus, $Z_3^4$ belongs to the LCS.

(b) Now, suppose that the current status quo is $Z_4^1$, and assume that all four players deviate and form the grand coalition. It follows from (A8) that $Z_j^4 \prec_i Z_i^4$ for any $i \in N$ and $j \neq 3, 4$, and that $Z_3^4 \prec_C Z_4^1$. Thus, we can obtain $Z_3^4 \prec_S B$, wherein $Z_3^4 \not\ll_S B$, only when $S$ contains a single player and $B = Z_3^4$. If a single player, say $4$, decides to deviate, $Z_3^4 \not\ll_{1,2,3,4} Z_4^1 \not\ll_4 \{(123), 4\} = Z_3^4$, then $Z_4^1 \not\ll_4 Z_3^4$, but $Z_4^1 \prec_{1,2,3,4} Z_3^4$. Therefore, if $Z = Z_1^4$, $V = Z_4^1$ and $S = \{1, 2, 3, 4\}$, then clearly $Z \not\ll_S V$, but we cannot find a coalition structure $B$, where $V = B = Z_4^1$ or $Z_4^1 \ll B$, such that $Z = Z_1^4 \not\ll_S B$, hence $Z_4^1$ cannot be stable.

(c) Next, suppose that the current status quo is $Z_2^1$, and assume that all four players deviate and form the grand coalition, $Z_2^1 = \{(12), 3, 4\} \not\ll_{1,2,3,4} Z_4^1$. Since it follows from (A8) that $Z_2^1 \not\ll_{1,2,3,4} Z_4^1$, a similar argument as the one used in (b) can be used to show that $Z_2^1$ cannot be stable.

(d) Assume that the current coalition structure is $Z_2^1$, and assume that all four players deviate and form the grand coalition, $Z_2^1 \not\ll_{1,2,3,4} Z_4^1$. Since it follows from (A8) that $Z_2^1 \not\ll_{1,2,3,4} Z_4^1$, a similar argument as the one used in (b) can be used to show that $Z_2^1$ cannot be stable.

(e) Suppose now that the current coalition structure is $Z_3^4$. It follows from (A8) that $Z_1^4 \ll_i Z_2^4 \ll_i Z_2^4 \ll_i Z_3^4$ for all $i$, hence for any coalition structure $V \not= Z_4^1$, obtained when some coalition $S$ deviates from $Z_3^4$, letting $B = Z = Z_3^4$ satisfies $V \ll B = Z_3^4$ and $Z = Z_3^4 \not\ll_S B = Z_3^4$. If $S = \{1, 2, 3, 4\}$ and $V = Z_4^1$, consider a following deviation by a player, 4,

$$Z_3^4 = \{(123), 4\} \not\ll_{1,2,3,4} Z_4^1 \not\ll_4 \{(123), 4\} = Z_3^4.$$  \hspace{1cm} (A11)

Then, $B = Z = Z_4^1$, $Z = Z_3^4 \not\ll_S B = Z_3^4$, while $Z_4^1 \ll B = Z_3^4$ because of (A11) and because $Z_4^1 \ll_4 \{(123), 4\} = Z_3^4$, which follows from (A8). Thus, $Z_3^4$ belongs to the LCS.

3. Lastly, suppose that $\alpha \leq 1.2516$. To show that $Z_4^1$ and $Z_2^4$ cannot be stable, one can use
Theorem 2

1. The grand coalition is always stable.

2. For $n \geq 3$, there is an $\alpha(n)$, defined by (5), such that any coalition structure of the form $\mathcal{Z}_{n-1}^n$, which contains an $(n-1)$-members coalition, is stable for $\alpha \leq \alpha(n)$.

3. For large $n$ there are values $\alpha_1$ and $\alpha_2$, $\alpha_1 < \alpha_2 < \alpha(n)$, such that: (a) when $\alpha < \alpha_2$, the outcome $\mathcal{Z}_{n-2,1,1}^n$ is stable; (b) when $\alpha < \alpha_1$, the outcome $\mathcal{Z}_{n-3,1,1,1}^n$ is stable.
In order to prove Theorem 2, we first need the following lemmas.

Lemma 2

1. For any \( k \leq \frac{n-1}{2} \), noncoalition members in the coalition structure \( \mathcal{Z}_k^n \) realize lower profit than the coalition members in the coalition structure \( \mathcal{Z}_{n-1}^n \).

2. For any \( k \leq \frac{n-1}{2} \), noncoalition members in the coalition structure \( \mathcal{Z}_k^n \) realize lower profit than the members of the grand coalition.

Proof of Lemma 2:

1. Recall that the profit for a coalition member in the basic coalition structure \( \mathcal{Z}_k^n \) increases with the size of the coalition. When \( n \leq 4 \), \( (n-1)/2 \leq 3/2 \), hence the condition \( k \leq (n-1)/2 \) implies \( k = 1 \), which corresponds to the coalition structure where all players act independently. Thus, the statement of Lemma 2 for \( n \leq 4 \) follows from Proposition 2. If we show that the statement holds for \( k = (n-1)/2 \) when \( n \geq 5 \) odd, the proof is complete.

By using expressions (3) and (4) from §3.2, one can show, although after tedious calculation, that

\[
\Pi_{\mathcal{Z}_{n-1}^n} - \Pi_{\mathcal{Z}_{n-1}^{n-1}} = K \left[ 8\alpha^2 n^2 \left( n^2 - 4n + 5 \right) + 2\alpha^3 n \left( 47n^3 - 119n + 93n + 3 \right) + \alpha^4 \left( 42\frac{1}{2} n^4 + 130n^3 - 915\frac{1}{2} n^2 + 896n - 80 \frac{3}{4} \right) + \alpha^5 \left( 26\frac{1}{2} n^4 - 60n^3 - 42\frac{1}{2} n^2 + 202\frac{1}{2} n - 106 - \frac{9}{8n} \right) + \alpha^6 \left( \frac{25}{4} n^4 + 9\frac{3}{4} n^3 - 119\frac{7}{16} n^2 + 143n - 54\frac{3}{8} - \frac{125}{4n} + \frac{217}{10n^2} \right) + \frac{17}{16} \left( 100n^3 - 424n^2 + 581n + 488 - \frac{166}{n} + \frac{8}{n^2} + \frac{61}{n^3} \right) \right],
\]

(A14)

where \( K \) is positive for \( \alpha > 0 \). It is easy to evaluate that the RHS of (A14) is positive for \( n \geq 5 \) and \( \alpha > 0 \), hence the first statement of the Proposition follows.

2. Follows from 1. because the coalition members in \( \mathcal{Z}_{n-1}^n \) always realize lower profit than the members of the grand coalition.

Next, denote by \( \mathcal{Z}_{n-k-1,k,1} = \{(12 \ldots n-k-1), (n-k \ldots n-1), n\} \) a coalition structure where players are divided into three coalitions, one containing \( n-k-1 \) players, the other containing \( k \) players, and one having a single member.
Lemma 3

1. In any coalition structure $Z_{n-k-1,k,1}^n$ that consists of three sets, one of which has only one member, a member of the largest coalition realizes lower profit than a coalition member in the coalition structure $Z_{n-1}^n$.

2. In any coalition structure $Z_{n-k-1,k,1}^n$ that consists of three sets, one of which has only one member, a member of the largest coalition realizes lower profit than a member of the grand coalition $Z_{n}^n$.

Proof of Lemma 3: Recall that Proposition 1 states that in any coalition structure, members of larger coalitions realize lower profit than the members of smaller coalitions. Therefore, in a coalition structure that consists of three sets, one of which has only one member, the members of the largest coalition realize the lowest profit, and it attains its highest value if, for $n$ odd, $k = (n - 1)/2$. Thus, if we prove the statement for the coalition structure $Z_{n-1,n-1,1}^{n-1}$, the statement of the Lemma follows. Note that each member of a coalition $C_{n-1}$ selects the same price and realizes the same profit, which maximizes the profit of $(n-1)/2$ members. Clearly, this profit is smaller than the one realized by a coalition member in the coalition structure $Z_{n-1}^n$, wherein the price is selected as to maximize the profit of all $n - 1$ members. Therefore, the first statement of the lemma holds. The second statement follows immediately from the first.

Denote by $Z_{n-i,i} = \{(12\ldots n - i), (n - i + 1\ldots n)\}$ a coalition structure where $n$ players are divided into two coalitions, one containing $i$ players, $C_i$, the other containing the remaining $n - i$ players, $C_{n-i}$.

Lemma 4 When the set of all players is divided into two coalitions, a player realizes the highest profit if he belongs to a one-member coalition.

Proof of Lemma 4: If we denote $D(i,n-i) = 4n^2(\alpha + 1) + 3i\alpha^2(n-i)$, then the profit for a player who belongs to the coalition $C_i$ can be written as $\Pi_{C_i}(n,i,\alpha) = A^2n[2n + \alpha(n+i)][n^2(\alpha + 1)(\alpha + 2) - i\alpha(n+i\alpha)]/D(i,n-i)^2$. Although $i$ is a discrete variable, let us suppose for a moment that it is continuous, and consider partial derivatives of $\Pi_{C_i}(n,i,\alpha)$ with respect to $i$. The second
partial derivative of $\Pi_C(n, i, \alpha)$ with respect to $i$ corresponds to

\[
\frac{\partial^2 \Pi_C(n, i, \alpha)}{\partial i^2} = \frac{\lambda^2 \alpha^2 n}{D(i, n - i)^3} \left[ 24n^2 i \alpha^4 + 3\alpha^3 (3n^3 + 22n^2 i - 9n^2 - 4i^3) + 2\alpha^2 n (23n^2 - 18ni - 9i^2) + 4n (16n^2 - 18ni + 9i^2) \right].
\]

One can evaluate that, for any $n \geq 2$ and $\alpha > 0$, the RHS of (A15) is positive, hence $\Pi_C(n, i, \alpha)$ is convex in $i$. It follows from Theorem 1 that, whenever the independent player in $Z_{n-1}$ realizes higher profit than he can generate in the grand coalition, each member of coalition $C_{n-1}$ realizes a lower profit than the independent player, $\Pi_{C_{n-1}} > \Pi_{C_{n-1}}$. Because $\Pi_C(n, i, \alpha)$ is convex in $i$, it further implies that $\Pi_{C_{n-1}} > \Pi_{C_{n-1}}$ for all $i = 2, \ldots, n - 1$.

**Proof of Theorem 2:**

1. Let us suppose that the grand coalition, $Z_n$, belongs to the LCS. Observe that, for $k \in \{1, \ldots, n - 1\}$, any deviation from $Z_n$ has the form $Z_n \rightarrow_{n-k+1, \ldots, n} Z_{n-k,k}$. Suppose that $k \geq n/2$, and consider the following sequence of deviations: $Z_n \rightarrow_{n-k+1, \ldots, n} Z_{n-k,k} \rightarrow_{n-k-k-1,1, \ldots, n-k} Z_{n-k} \rightarrow_{n-k,1, \ldots, n} Z_n$, where $Z_{n-k} = \{(12 \ldots n-k), 1, \ldots, 1\}$. Let $Z = B = Z_n, V = Z_{n-k,k}$, and $S = \{n-k+1, \ldots, n\}$. Now, it is true that $Z_{n-k,k} \prec V \preceq B = Z_n$, because:

   (a) $Z_{n-k,k} \prec Z_n$ follows from the fact that the grand coalition realizes the highest total profit, and that a member of the larger coalition in $Z_{n-k,k}$ realizes lower profit than a member of the smaller coalition;

   (b) $Z_{n-k,k-j,1,\ldots,1} \prec_{n-j-1} Z_{n-k}$, for all $j = 0, \ldots, k - 1$ follows from Lemma 3 and Proposition 1;

   (c) $Z_{n-k} \prec_{1,\ldots,n} Z_n$ follows from Lemma 2, the fact that $n - k \leq n/2$ for $k \geq n/2$, and the fact that the profit for a coalition member in $Z_j$ increases with the size of the coalition.

In addition, $Z_n = Z \not\prec_S B = Z_n$. Note that, when $k < n/2$, the analysis still holds after replacing the deviation of players $n-k+1, \ldots, n$ by a deviation of players $1, \ldots, n-k$. Thus, the grand coalition always belong to the LCS.

2. Proof of item 2 is similar to that of item 1 and is therefore omitted. It uses Proposition 1 and Lemmas 2, 3, and 4.
3. Consider the coalition structure $Z_{n-2,1,1}^n$ (and $Z_{n-3,1,1,1}^n$), where $n - 2$ ($n - 3$) players form a coalition, and the remaining two (three) players, say $i$ and $j$ ($i, j,$ and $k$) remain independent. Using exactly the same analysis as in Proposition 3, we can show that there exists $\alpha^1 < \alpha^2 < \alpha(n)$ such that:

(a) $\Pi_i Z_{n-2,1,1}^n > \Pi_i Z_n^n$ when $\alpha < \alpha^2$;

(b) $\Pi_i Z_{n-3,1,1,1}^n > \Pi_i Z_n^n$ when $\alpha < \alpha^1$; and

(c) $\Pi_i Z_{n-3,1,1,1}^n < \Pi_i Z_{n-2,1,1}^n$ when $\alpha > \alpha^1$.

Now consider all possible outcomes that are obtained by a single defection from $Z_{n-2,1,1}^n$. They are: (i) $Z_{n-2,1,1}^n \rightarrow Z_n^n$; (ii) $Z_{n-2,1,1}^n \rightarrow Z_{n-k,k}^n$; (iii) $Z_{n-2,1,1}^n \rightarrow Z_{n-k-l,k,l}^n$; and (iv) $Z_{n-2,1,1}^n \rightarrow Z_{n-k-l-1,k,l}^n$. Note that (ii) and (iii) can be further divided into sub cases depending on whether either $i$ or $j$ are involved in the defection.

It is easy to show that (i) is deterred by considering the sequence $Z_{n-2,1,1}^n \rightarrow Z_n^n \rightarrow Z_{n-2,1,1}^n$. To prove (ii), we first show, exactly imitating the proof of Proposition 4, that there exists $\psi > \alpha^1$ such that for $\alpha < \min\{\psi, \alpha^2\}$ coalition outcome with exactly two coalitions that are not in the LCS is indirectly dominated by a basic coalition structure. This shows that (ii) is deterred. (iii) and (iv) can be shown exactly using chains as in (1) and (2). This proves part (3a). The proof of (3b) is similar and uses (b) and (c) above. 

**Proposition 7** For $n < 4$, the grand coalition is the unique stable outcome.

**Proof of Proposition 7:** When there are no coalitions, i.e. players operate independently, the expected profit can be written as

$$
\pi(p_i, q_i) = E \left[ (p_i - c)(D_i(p_i) + \varepsilon) - \left\{ (p_i - c) [D_i(p_i) + \varepsilon - q]^+ + (c + h) [q - (D_i(p_i) + \varepsilon)]^+ \right\} \right],
$$

where

$$D_i(p_i) = A - (1 + \alpha)p_i + \frac{\alpha}{n} \sum_{j=1}^n p_j.
$$

Instead of working with the purchasing quantity $q_i$, we use the transformation $\lambda_i = q_i - D_i(p_i)$. Thus,

$$
\pi(p_i, \lambda_i) = (p_i - c)(D_i(p_i) + \mu) - \left\{ (p_i - c) \int_{\lambda_i}^{\infty} (u - \lambda_i)f(u)du + (c + h) \int_{-\infty}^{\lambda_i} (\lambda_i - u)f(u)du \right\}.
$$
The equilibrium prices and quantity are given by
\[ p^* = \frac{A + (1 + \frac{2}{3} \alpha) c + \mu - \int_{\lambda_i}^{\infty} (u - \lambda^*) f(u) du}{2(1 + \frac{1}{3} \alpha)}, \quad F(\lambda^*) = \frac{p^* - c}{p^* + h}, \]
where \( F \) and \( f \) are the cumulative distribution and density of \( N(\mu, \sigma) \), respectively.

When all players form the grand coalition, they face a demand \( A + p + \varepsilon \). In this case, calculating the optimal \((p^*, \lambda^*)\) yields
\[ p^* = \frac{A + c + \mu - \int_{\lambda_i}^{\infty} (u - \lambda^*) f(u) du}{2}, \quad F(\lambda^*) = \frac{p^* - c}{p^* + h}. \]
Finally, when two players form an alliance (which corresponds to the structure \( Z^3_2 \)), denote the price set by the coalition at equilibrium as \( p_C^* \), and the price set by the lone non-member as \( p_C^* \). Then, we have
\[ p_C^* = \frac{A + (1 + \frac{2}{3} \alpha) \frac{6 + 3 \alpha}{6 + 5 \alpha} c + \mu - \int_{\lambda_i}^{\infty} (u - \lambda_C^*) f(u) du}{2}, \quad F(\lambda_C^*) = \frac{p_C^* - c}{p_C^* + h}, \]
\[ p_C^* = \frac{A + (1 + \alpha) \frac{2 + \alpha}{3 + 2 \alpha} c + \mu - \int_{\lambda_i}^{\infty} (u - \lambda_C^*) f(u) du}{2}, \quad F(\lambda_C^*) = \frac{p_C^* - c}{p_C^* + h}. \]
The remaining analysis follows the steps similar to those in Section 3.

**Proposition 8** Suppose \( n > 0 \), and \( \mu, \sigma > 0 \). Furthermore, suppose that the parameters are such that the price charged by the grand coalition, \( p^* \), satisfies inequality \( p^* > h + 2c \). Then, there exists \( \alpha^* \) such that the grand coalition is uniquely stable when \( \alpha \geq \alpha^* \).

**Proof of Proposition 8:** In any structure, if player \( i \) sets \((p_i, \lambda_i)\), his profit is given by
\[ \Pi_i(s_i, \lambda_i) = s_i(A + c - s_i) - (h + c) \int_{-\infty}^{\lambda_i} (\lambda_i - u)f(u)du - s_i \int_{\lambda_i}^{\infty} (u - \lambda_i)f(u)du + \]
\[ \alpha s_i \left(-s_i + \frac{1}{n} \sum_{j=1}^{n} s_j \right), \]
where \( s_i = p_i - c \). We write \( \Pi_i^{Z}(s_i, \lambda_i) = B_i(s_i, \lambda_i, Z) + L_i(s_i, \lambda_i, Z) \), where
\[ B_i(s_i, \lambda_i) = s_i(A + c - s_i) - (h + c) \int_{-\infty}^{\lambda_i} (\lambda_i - u)f(u)du - s_i \int_{\lambda_i}^{\infty} (u - \lambda_i)f(u)du \]
\[ L_i(s_i, \lambda_i) = \alpha s_i \left(-s_i + \frac{1}{n} \sum_{j=1}^{n} s_j \right). \]
This expression is in free form. In a particular coalition structure $Z = \{Z_1, \ldots, Z_k\}$, players in coalition $Z_k$ will set the same price and inventory levels. Thus, even though formally there is a dependence on $Z$, we drop this from the expressions. The regularity conditions imply that for a fixed vector $(\lambda_1, \ldots, \lambda_k)$, $B_i$ and $L_i$ are unimodal in $(s_1, \ldots, s_k)$ and vice versa. Clearly, they are differentiable and hence continuous in the variables as well. The total profit of all players under any coalitional structure equals $\Pi^Z_{Total} = \sum_{i=1}^n \Pi_i(s_i) = \sum_{i=1}^n B_i(s_i) + \sum_{i=1}^n L_i(s_i)$. Note that

\[
\sum_{i=1}^n L(s_i) = \alpha \left( - \sum_{i=1}^n s_i^2 + \frac{1}{n} \sum_{j=1}^n \sum_{k=1}^n s_j s_k \right) = \alpha \left( - \frac{n-1}{n} \sum_{i=1}^n s_i^2 + \frac{1}{n} \sum_{j \neq k}^n 2s_j s_k \right) = -\frac{\alpha}{n} \sum_{j \neq k} (s_j - s_k)^2 \leq 0
\]

and

\[
\frac{\partial}{\partial s_i} \sum_{j=1}^n B(s_j) = \frac{\partial B(s_i)}{\partial s_i} = A + \mu - c - 2s_i - \int_{\lambda_i}^{\infty} (u - \lambda_i) f(u) du - (h + c) \frac{\partial}{\partial s_i} \int_{-\infty}^{\lambda_i} (\lambda_i - u) f(u) du
\]

Therefore, in any equilibrium outcome $\lambda_i = F^{-1}(s_i/(s_i + c + h))$ implies that

\[
\frac{\partial}{\partial s_i} \sum_{j=1}^n B(s_j) = \frac{\partial B(s_i)}{\partial s_i} = A + \mu - c - 2s_i - \int_{\lambda_i}^{\infty} (u - \lambda_i) f(u) du
\]

and $\Pi^Z_{Total} = \sum_{i=1}^n B_i(s_i) + \sum_{i=1}^n L_i(s_i)$ achieves its maximum at

\[
s_i^* = \frac{A + \mu - c - \int_{\lambda_i}^{\infty} (u - \lambda_i^*) f(u) du}{2}, \quad \lambda_i^* = F^{-1} \left( \frac{s_i^*}{s_i^* + c + h} \right)
\]

for all $i = 1, \ldots, n$ when $Z = N$, the grand coalition.

Now, when $p^* > h + 2c$, we can show using continuity that there exists $\alpha^* > 0$ such that for $\alpha > \alpha^*$, $\Pi_1^{Z_1} = B_1(s_1^*, \lambda_1^*, Z_{n-1}) + L_1(s_1^*, \lambda_1^*, Z_{n-1}) < \Pi^N_{Total}/n$. This immediately implies that the grand coalition is uniquely stable.

**Proposition 9** Let $Z = \{Z_1, \ldots Z_l\}$ be a coalition structure with at least two nonempty coalitions.

1. $\sum_{i=1}^n \Pi_i^Z < \sum_{i=1}^n \Pi_i^N$.

2. If $i \in Z_k$, $j \in Z_m$, and $|Z_k| > |Z_m|$, then $\Pi_i^Z < \Pi_j^Z$.

3. Suppose $n > 2$, and $i \in Z_k$ leaves the coalition, thereby changing the coalition structure to $Z' = \{Z_1, \ldots, Z_k \setminus \{i\}, \{i\}, \ldots, Z_l\}$. Then $\Pi_i^{Z'} \geq \Pi_i^Z$.  

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Proof of Proposition 9:

1. At equilibrium every coalition charges the same wholesale price. It is easy to see that, if there are \( l \) coalitions, \( w = a/(l+1) \cdot b \) and \( Q = a/[2(l+1)] \), which immediately implies that each coalition makes \( a/[2b(l+1)^2] \). Since \( J(l) = 1/(l+1)^2 \) is decreasing in \( l \), the grand coalition is the efficient outcome and this proves item 1.

2. Item 2 trivially follows from the fact every coalition makes the same profit and members divide the profit equally (recall that \( c_i = 0 \)).

3. To prove item 3, note that \( Z \) has \( l \) coalitions and \( Z' \) has \( l+1 \) coalitions, and

\[
\Pi_i^{Z'} = \frac{a^2}{2b(l+2)^2} \geq \frac{1}{2} \left[ \frac{a^2}{2b(l+1)^2} \right] \geq \Pi_i^{Z}.
\]

Proposition 10

1. The grand coalition is always stable.

2. For \( n < 5 \), the grand coalition is uniquely stable.

3. For large \( n \), no basic coalition structure (including \( Z_1^n \)) is stable.

4. For large \( n \), there is at least one equal-sized stable coalition structure other than \( N \).

Proof of Proposition 10:

1. Consider \( N \rightarrow_S \mathcal{X} \), a deviation from the grand coalition by a set of players \( S \). Consider the following sequence \( N \rightarrow_S \mathcal{X} \rightarrow_{S_1} \mathcal{X}_1 \rightarrow ... \rightarrow_{S_k} \mathcal{X}_k \rightarrow_{S_{k+1}} Z_1^n \rightarrow_N N \), where at each step before the last step a single member splits from the largest coalition (ties broken arbitrarily) to create the new outcome. In any outcome \( \mathcal{X}_1 \) we know that the members of the largest coalition must earn lower than in the grand coalition. Thus \( \mathcal{X} \ll N \) and \( N \not\ll_S N \). This proves item 1.

2. We first note that in any coalition structure that is nonsymmetric (i.e. has unequal coalition sizes), if the players in the largest coalition make less than they would in \( Z_1^n \), then the grand
coalition is the unique member of the set of outcomes that indirectly dominate all outcomes. The proof of this observation is similar to Proposition 4 and uses Proposition 9.

For \( n = 1, 2 \), the result is trivial. For \( n = 3 \), the profit of a coalition member in \( Z^3_2 \) is \( a^2/36b \), while in \( Z^3_1 \) each player makes \( a^2/32b \). For \( n = 4 \), the profit of a coalition member in \( Z^4_3 \) and in \( Z^4_1 \) is \( a^2/(54b) \), which is lower than the allocation \( a^2/(50b) \) achieved in \( Z^4_1 \). This proves \( N \) is uniquely stable.

3. Next, to prove that for any \( n \), \( Z^n_1 \) and no basic coalition structures are in the LCS, we use an alternate definition of the LCS. The LCS is the fixed point of a suitably defined map which we have used in earlier proofs. This implies that the LCS can be written as \( \bigcap_{k=0}^n Y_k \), where \( Z = Y_0 \), and inductively \( Z \in Y_{k-1} \) belongs to \( Y_k \) if and only if \( \forall X \) and \( S \) such that \( Z \rightarrow S X \), \( \exists W \in Y_{k-1} \), where \( W = X \) or \( X \ll W \) such that \( \Pi^S_X < \Pi^S_Y \) does not hold.

Note that in \( Z^n_1 \) each player makes \( a^2/(2b(n+1)^2) \), which is smaller than what any player makes in \( N \). This alone does not prove that \( Z^n_1 \) is not in the LCS. Define, for any \( k \), \( J_k \) as the minimum profit obtained by coalitions when an equal-sized coalition structure with \( k \) members is formed. Define

\[
T_k = \min \left\{ \frac{a^2}{2b(k+1)^2 \left\lfloor \frac{n}{k} \right\rfloor}, \frac{a^2}{2b(k+1)^2 \left\lceil \frac{n}{k+1} \right\rceil} \right\}
\]

Note that \( T_k \geq \frac{a^2}{2b(n+1)^2} \forall k = 1, 2, \ldots n \) when \( n \) is large.

Consider now any basic coalition structure in which there are totally \( k \) coalitions, \( Z^n_{n-k+1} \). Players in \( Z^n_1 \) make less than those in largest coalition in such a basic coalition structure if and only if

\[
\frac{a^2}{2b(n+1)^2} < \frac{a^2}{2b(k+1)^2(n-k+1)}.
\]

Note that \( f(x) = (x+1)^2(n-x+1), x \leq n \) is concave and is maximized at \( x = \frac{2n+1}{3} \), with

\[
f \left( \frac{2n+1}{3} \right) = (5n+4)^2(n-2) > (n+1)^2.
\]

Thus \( Z^n_1 \) does not indirectly dominate \( Z^n_{n-k+1} \) and single defections from \( Z^n_{n-k+1} \) are not deterred. Further since, \( Z^n_1 \rightarrow_N N \) is not deterred, both \( Z^n_{n-k+1} \) and \( Z^n_1 \) are not elements of \( Y_k \) for some \( k \geq 1 \). This proves item 3.
4. To show that \( \exists U \in \text{LCS}, U \neq N \), start with \( Y_0 = \{ \mathcal{X} \in \mathcal{Z}, \mathcal{X} \neq N \text{ and } \mathcal{X} \text{ has "equal" sized coalitions } \} \), where "equal" is defined as before. Using the same iterative process and arguments as in 3, we can show that \( \bigcap Y_k \neq \emptyset \). This implies item 4. 

**Proposition 11**

1. When \( n = 3 \), the grand coalition is the unique absorbing state of the EPCF.

2. When \( n = 4 \), the grand coalition is the unique absorbing state if \( \alpha > 1.864 \); if \( \alpha \leq 1.864 \), then any coalition structure with an alliance of three players is also absorbing.

**Proof of Proposition 11:** To prove item 2, let us first define \( p \) as follows:

\[
\{(123), 4\} \to_3 \{(12), 3, 4\}, \{(234), 1\} \to_4 \{(23), 1, 4\},
\]

\[
\{(134), 2\} \to_1 \{(34), 1, 2\}, \{(124), 3\} \to_2 \{(14), 2, 3\},
\]

and \( p(\mathcal{Z}, \mathcal{Z}_1^4) = 1 \) for all remaining coalition structures \( \mathcal{Z} \). We want to show that this is an EPCF with absorbing state at the grand coalition. It is easy to verify that players always benefit by deviating from any coalition structure which is not in the LCS. This is also true for \( \mathcal{Z}_2^3 \): since \( v(\mathcal{Z}_1^4, p) = \Pi_{C}^{\mathcal{Z}_1^4} + \delta v(\mathcal{Z}_1^4, P) \), it follows that \( v(\mathcal{Z}_1^4, p) = \Pi_{C}^{\mathcal{Z}_1^4}/(1 - \delta) \), while \( v(\mathcal{Z}_2^2, p) = \Pi_{C}^{\mathcal{Z}_2^2} + \delta v(\mathcal{Z}_2^2, p) \). Thus, \( v(\mathcal{Z}_1^4, p) - v(\mathcal{Z}_2^3, p) = \Pi_{C}^{\mathcal{Z}_1^4} - \Pi_{C}^{\mathcal{Z}_2^3} > 0 \). Lastly, we need to show that player \( i \) has an incentive to deviate from \( \{(ijk), l\} \) to \( \{jki\} \), which is equivalent to \( v_i(\{(ijk), l\}, p) \leq \Pi_{C}^{\mathcal{Z}_1^4} + \delta v_i(\mathcal{Z}_1^4, p) \). This is true if \( \Pi_{C}^{\mathcal{Z}_1^4} + \delta \Pi_{C}^{\mathcal{Z}_1^4} + \delta^2/(1 - \delta) \Pi_{C}^{\mathcal{Z}_1^4} \leq \Pi_{C}^{\mathcal{Z}_2^3} + \delta/(1 - \delta) \Pi_{C}^{\mathcal{Z}_2^3} \), which is satisfied for \( \delta > (\Pi_{C}^{\mathcal{Z}_1^4} - \Pi_{C}^{\mathcal{Z}_2^3})/(\Pi_{C}^{\mathcal{Z}_1^4} - \Pi_{C}^{\mathcal{Z}_2^3}) \). It is easy to evaluate that the RHS of this inequality never exceeds 0.25.

Next, we want to construct a PCF which is an EPCF with absorbing state at \( \{(123), 4\} \). For \( i, j, k \in \{1, 2, 3\}, i \neq j \neq k \neq i \), let us define PCF as follows:

\[
\{(1234)\} \to_4 \{(123), 4\}, \quad \{(123), 4\} \to \{(123), 4\}, \quad \{(ij4), k\} \to_4 \{(ij), k, 4\},
\]

\[
\{(ij), k, 4\} \to_{1, 2, 3} \{(123), 4\}, \quad \{(ij), k, 4\} \to_4 \{(1, 2, 3, 4)\}, \quad \{(ij), (k4)\} \to_4 \{(ij), k, 4\},
\]

\[
\{1, 2, 3, 4\} \to_{1, 2, 3} \{(123), 4\}.
\]

Again, it is easy to verify that players always benefit by deviating from any coalition structure which is not in the LCS, and that player 4 benefits when he deviates from the grand coalition. Next, consider a coalition structure wherein 4 is a member of a three-player coalition, say \( \{(124), 3\} \). We need
to show that 4 has an incentive to deviate from \{(124), 3\}, which is equivalent to \(v_4(\{(124), 3\}, p) \leq \Pi_{C}^{Z_4^1} + \delta v_4(\{(123), 4\}, p)\). Thus, we must have \(\Pi_{C}^{Z_4^1} + \delta \Pi_{C}^{Z_4^1} + \delta^2 / (1 - \delta) \Pi_{C}^{Z_4^1} \leq \Pi_{C}^{Z_4^1} + \delta / (1 - \delta) \Pi_{C}^{Z_4^1}\), which is satisfied for \(\delta > (\Pi_{C}^{Z_4^1} - \Pi_{C}^{Z_4^1}) / (\Pi_{C}^{Z_4^1} - \Pi_{C}^{Z_4^1})\). It is easy to evaluate that the RHS of this inequality never exceeds 0.25 when \(\alpha \leq 1.864\). Similarly, 4 has an incentive to deviate from \{(12), (34)\} if \(\delta > (\Pi_{C}^{Z_4^2}, 2 \Pi_{C}^{Z_4^2}, 2) / (\Pi_{C}^{Z_4^3} - \Pi_{C}^{Z_4^3})\). It is easy to evaluate that the RHS of this inequality never exceeds 0.18 when \(\alpha \leq 1.864\). In a similar way, it can be shown that the remaining coalition structures of the form \(Z_{3}^{4}\) are absorbing.

\[\text{Theorem 3}\]

1. In the substitute model with large \(n\), \(\exists \alpha > 0 \text{ and } \Delta_1^* > 0 \text{ such that for all } 1 > \delta > \Delta_1^*\), (a) \(N \notin \text{EP}(n, \alpha)\), and (b) \(Z_k^n \in \text{EP}(n, \alpha)\), where \(Z_k^n\) is a basic coalition structure.

2. For the substitute model with uncertain demand, if \(p^*\) is such that \(p^* > h + 2c\), then there exists \(\alpha^*\) such that the grand coalition is the unique absorbing state of the EPCF.

3. In the assembly model with large \(n\), \(\exists \Delta_2^*\) such that for \(1 > \delta > \Delta_2^*\), \(N \subset \text{EP}(n) \subset N \cup \{\text{equal-sized coalition structures}\} \setminus \{Z_1^n\}\).

\[\text{Proof of Theorem 3:}\]

1. Let \(|Z| = t\), and let \(m\) and \(M\) be minimum and maximum single period payoffs to any player, respectively. Choose \(\Delta_1^* \in (0, 1)\) such that for any two states \(X, Y\), and any player \(i\) with \(u_i(X) > u_i(Y)\), we have

\[u_i(X) > [1 - (\Delta_1^*)^t]M + (\Delta_1^*)^t u_i(Y), \text{ and } u_i(X)(\Delta_1^*)^t + [1 - (\Delta_1^*)^t]m > u_i(Y).\]

Consider any \(\delta \in (\Delta_1^*, 1)\) and fix an arbitrary absorbing deterministic EPCF with absorbing states \(X \subseteq Z\). Let \(Z \in Z\), and \(Y \in F_S(Z)\) for some fixed coalition \(S\). Let \(Y_1, Y_2, \ldots, Y_r\) be the subsequent PCF path, where \(Y_1 = Z\), and \(Y_r = Y\). Then, there is an \(S_i\) such that \(Y_{i+1} \in F_{S_i}(Y_i)\) and \(v_i(Y_{i+1}) \geq v_i(Y_i)\). It is possible now to chose \(\delta\) and \(n\) such that a basic coalition is in the LCS and

\[v_j(Y_i) \geq \Pi^N / n \text{ for } j \in S_i.\]
Now, for this $\delta$ and $n$, we can also show $v_i(\mathcal{Y}) \geq v_i(\mathcal{Y}_j)$ and $u_i(\mathcal{Y}) \geq u_i(\mathcal{Z})$. Define $f : \mathcal{Z} \rightarrow \mathcal{Z}$ as $f(\mathcal{Z}) = \{\mathcal{X} \in \mathcal{Z} : \forall \mathcal{Y} \in F_S(\mathcal{X}) \exists \mathcal{Z} \in \mathcal{Z} \text{ such that } \mathcal{Y} = \mathcal{Z} \text{ or } \mathcal{Y} \ll \mathcal{Z} \text{ and } u_i(\mathcal{X}) \geq u_i(\mathcal{Z}) \text{ for some } i \in S\}$. It is possible to show $\mathcal{Z} \subseteq f(\mathcal{Z})$. Choose $\delta$ and $n$ as above so that $\mathcal{Z} \subseteq f(\mathcal{Z}) \subset LCS \cap \{ \text{basic coalition structure } \}$, where the second inclusion follows from Proposition 4 and (A15).

2. The result follows directly from Proposition 8, because $EP(n)$ is a subset of the LCS.

3. Proof is similar to that of item 1. However, note the following. There exists an equal-sized coalition structure and $\delta$ and $n$ such that the equal-sized coalition structure is in the LCS and (A15) is true. This shows $N \subset EP(n)$. Note that when two coalitions merge from an equal-sized coalition structure, the merger is not myopically beneficial to the merging coalitions. Thus, $N$ cannot be a unique absorbing state, and $EP(n) \subset N \cup \{\text{equal-sized coalition structures}\}\{\mathcal{Z}_1^n\}$.