A collaborative decentralized distribution system with demand updates

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29.01.2008.

Abstract

In this paper, we consider a distribution system that consists of a manufacturer, a warehouse (or a distribution center), and \( n \) retailers. At the time their orders are placed, the retailers know their demand distribution but do not know the exact value of the demand. After certain production and transportation lead time elapses, the orders arrive at the warehouse. During this time, the retailers can update their demand forecasts.

We first focus on cooperation among the retailers – the retailers coordinate their initial orders and can reallocate their orders in the warehouse after their forecasts are updated. We consider two contracting schemes between the manufacturer and the retailers, and show the nonemptiness of the core of the associated cooperative game between the retailers. Next, we analyze the impact that cooperation and non-cooperation of the retailers has on the manufacturer’s profit. Finally, we focus on coordination of the entire supply chain through buy-back contracts, which are known to coordinate the systems with full information. Our model differs from the traditional buy-back contract, in which the return of inventories occurs only after all uncertainty about demand has been resolved, because inventory is returned at the warehouse (before the true demand has been revealed). We show that buy-back contracts, in general, cannot coordinate the distribution system, unless the buy-back price equals the salvage value at the warehouse, and that the gap between the centralized solution and the decentralized output can be arbitrarily large.

Most of our analysis assumes that the retailers in a coalition do not share their information. However, we also analyze the impact of information sharing in a setting with buy-backs.

Keywords: inventory centralization, coordination, game theory, core.
1 Introduction

In today’s global world, it is a common practice to supply several markets at dispersed locations with production from another market. While the manufacturers benefit from lower production costs and can offer their products at lower prices, the retailers observe longer lead times. Hence, they need to place their orders earlier, and consequently they face higher uncertainty of the demand that will be satisfied by those orders.

One way of dealing with high demand uncertainty is the pooling of inventory. Inventory pooling is known to reduce the risk of mismatching demand and supply, and thus it increases profit. Traditionally, the retailers physically consolidate their inventories to benefit from risk pooling. Sometimes, however, it may not be possible to implement physical pooling because of the dispersed locations of the retailers and customers’ unwillingness to wait until the products are delivered from the central warehouse. In such situations, the retailers have to keep stock at their local facilities in order to satisfy their customers. However, if the retailers can obtain better information about future demand while their orders are on the way, they can still benefit from risk reduction by reallocating their orders when they arrive at the facility where the reallocation can take place after demand information update (e.g., port, warehouse, crossdocking station, etc.). For simplicity, we hereafter refer to such location as a warehouse.

In this paper, we consider a distribution system that consists of a manufacturer, a warehouse, and $n$ retailers. Each retailer sells an identical product, made by the manufacturer. Due to long production and transportation lead times, they have to place their orders without knowing the exact value of the demand, but they have information about its distribution. The orders arrive at a warehouse after some production and transportation lead time elapses, during which time the retailers update their demand forecasts. We first focus on cooperation among the retailers. The retailers jointly place their orders and can reallocate the amount that they ordered (by taking into consideration the updated forecasts). This possibility of reallocation is taken into account when the initial orders are placed. We investigate a cooperative game between the retailers that abstracts away the details of mechanisms leading to cooperation but allows us to analyze the problem of profit division in more detail. We study two contracting schemes between the retailers and the manufacturer – namely, the wholesale-price contract and the buy-back contract. For each contract, we prove that there exists a stable allocation of joint profit among the retailers by showing nonemptiness of the core of the associated cooperative game. In addition, we show how a core allocation may be computed. Hence, when such an allocation is used, cooperating retailers prefer the grand coalition to being
in any other alliance. We extend this case by considering cooperation among the retailers with information sharing, in which the retailers share their demand signals among themselves, and show core nonemptiness for this case as well.

Following this, we concentrate on the relationship between the manufacturer and the retailers. Under a wholesale price contract, the manufacturer’s profit is determined by the total retailers’ order quantity and the wholesale price. Since the retailers’ stocking quantities depend on their membership in different coalitions, the manufacturer’s profit is affected by the structure of the retailers’ alliances, and this impact might be either positive or negative. It can be shown that when demand is skewed left or the understocking cost is high, the manufacturer’s profit decreases as a result of the retailers’ cooperation. However, the manufacturer may benefit from their cooperation when the demand follows a right-skewed distribution (and centralization reduces the risk associated with a larger order size) or when the overstocking cost is high, which consequently may result in a higher service level.

Unlike the wholesale price contract, a buy-back contract allows the manufacturer to offer a buy-back opportunity to the retailers after the demand update (when the orders arrive at the warehouse), but before the demands are realized at the retailers’ markets. By offering a buy-back price, the manufacturer bears a part of the retailers’ risk, and thus encourages them to order higher quantities. Although the manufacturer enjoys higher revenues due to higher order quantities, returns may raise the manufacturer’s expenses. We assume that the manufacturer can salvage items left at the warehouse. For a manufacturer who serves several markets, it might be possible to direct some inventory left over in one market to other markets that have non-overlapping selling periods. We incorporate this possibility into our model by defining a positive salvage value at the warehouse, which can only be realized by the manufacturer. Thus, when the manufacturer has the possibility of salvaging unsold items, it can always increase its profit by offering a buy-back price that is equal to the salvage value. Similar to the wholesale price case, the retailers’ cooperation may have either a positive or negative impact on the manufacturer’s profit under buy-back contracts.

It is known in the literature that centralization increases the profits in distribution systems and that buy-back contracts are capable of coordinating the system with a single manufacturer and a single retailer. In addition, any allocation of joint profit can be achieved by an appropriate buy-back contract. These results can be extended to our model with multiple retailers if the retailers all face the same difference between the selling price and the transportation cost, and if the demand signal reveals perfect demand information. However, if the demand signal is not perfect, the buy-back contract fails to reach the profit level of the centralized system (even if it induces optimal order quantities) whenever the buy-back cost differs from the salvage value at the warehouse. In an
example, we show that this gap can be arbitrarily large.

The remainder of the paper is organized as follows. Section 2 contains a brief literature review. In Section 3, we give preliminaries on cooperative game theory and introduce some concepts that we subsequently use in the paper. In Section 4, we present our newsvendor model with updated demand information. In Section 4.2, we consider the centralized system, which is subsequently used as a benchmark. We analyze the decentralized systems with the wholesale price contract and with the buy-back contract in Section 4.3 and 4.4, respectively. In these sections, we first focus on the retailers’ cooperation and then investigate its effect on the manufacturer’s profit. Following this, we compare the two contracts. In Section 5, we analyze coordination of our distribution system with updated demand information through buy-back contracts. Most of our analysis assumes that the retailers in a coalition do not share their information. In Section 6, we analyze the impact of information sharing in a setting with buy-backs. We conclude in Section 7 with a summary and discussion of the results. The proofs are given in a technical appendix.

2 Literature

In the inventory literature, many authors have analyzed the effects of inventory centralization – namely cost reduction and profit increase. Among others, Eppen (1979), Eppen and Schrage (1981), Chen and Lin (1989), Chang and Lin (1991), and Cherikh (2000) show this effect in different inventory settings. All of the above-mentioned studies assume single ownership of the system. Individual firms, however, are especially interested in what they can get for themselves from inventory centralization. Several papers have investigated how to share the extra profit from inventory centralization among the individual firms from the viewpoint of cooperative game theory. For instance, Hartman et al. (2000) study models with multiple newsvendors, focusing especially on the core of associated cooperative games. They show the non-emptiness of the core of these games under some restrictive assumptions on demand distributions. Müller et al. (2002) and Slikker et al. (2001) independently develop a stronger result, showing that newsvendor games have a non-empty core regardless of the demand distribution. Hartman and Dror analyze cooperation through inventory centralization in a newsvendor setting in several papers. Hartman and Dror (2003) study the cost game among the retailers with normally distributed and correlated individual demands. After observing that the value of a coalition is a function of covariance matrix, they introduce a greedy optimization procedure manipulating the correlations to minimize costs. Furthermore, they show that nucleolus gives payoffs in the core of the corresponding game at each step of the procedure. Hartman and
Dror (2005) show that the core of a cost game with non-identical holding and penalty costs might be empty, and derive the conditions under which such a game will be subadditive. They then focus on the cost allocation game based on demand realization and show that the core of such a game might be empty even for identical cost parameters for the retailers. Slikker et al. (2005) and Özen et al. (2004) consider several extensions of simple newsvendor models. Slikker et al. (2005) introduce non-identical selling and purchasing prices and transshipment costs and they show that newsvendor games with transshipments have a non-empty core. Similarly, Özen et al. (2004) show that games associated with newsvendor models with warehouses have a non-empty core. Our paper extends the latter work by looking at demand forecast updates as well as buy-back opportunities. Chen and Zhang (2006) show how core allocations can be calculated in inventory games by using stochastic programming duality approach. We use their approach to calculate core allocations in our model.

Another type of inventory centralization, this time through transshipments, is considered by Anupindi et al. (2001). In their model, the retailers may transship the excess inventory from one location to satisfy the excess demand in another location. They assume that the retailers place their orders competitively and then make cooperative transhipment decisions. They derive a profit-allocation mechanism, based on dual prices in the transshipment problem, that leads to core allocations and induces equilibrium order quantities that are optimal for the entire system. Granot and Sošić (2003) extend their model by considering an intermediate stage in which the retailers decide how much of their excess inventory/demand they want to share with others. After showing that the mechanisms based on dual prices might not induce the retailers to share their entire excess inventory/demand, they show that the Shapley value and the fractional rule have this property. Sošić (2006) analyzes a similar cooperative game with transshipments and shows that Shapley value allocations are stable in a farsighted sense but not necessarily stable in a myopic sense.

The papers mentioned above study the impact that cooperation has on the retailers. However, the manufacturer is not isolated from the effects of cooperation or competition at the retailers’ level. Dong and Rudi (2004) study the effect of transshipment on the manufacturer and the retailers under both exogenous and endogenous wholesale price contracts. They assume identical-cost retailers and single ownership. After providing analytical results for order quantities and the profits in the exogenous wholesale price case, they focus on the model with a price-setting manufacturer. They show that risk pooling through transshipment makes the retailers’ order quantities less sensitive to the wholesale price, which, in general, results in higher manufacturer’s profit and lower retailers’ profits. Zhang (2005) extends the results of Dong and Rudi (2004) in the exogenous wholesale price case by allowing general demand distributions, under the assumption that all retailers face
identical demand distributions. He utilizes stochastic comparison techniques to study the impact of transshipment on retailers’ order quantities and the resulting profits. Zhao et al. (2005) study a continuous-review infinite-horizon distribution system with two retailers. In this case, using a base-stock and threshold-rationing policy, the retailers share their inventories through transshipments, but determine their policy parameters non-cooperatively. The authors investigate the existence of equilibrium strategies for the retailers and then perform an extensive numerical study that analyzes the effects of manufacturer’s incentives and subsidies on retailers’ inventory-sharing behavior. Anupindi and Bassok (1999) study the effect of stock centralization in a one manufacturer-two retailers distribution system with market search. The retailers can either cooperate (to benefit from centralization of stocks) or compete (keeping their individual stocks). The authors show that the manufacturer may not always benefit from centralization. Whether it will happen or not depends upon the demand distribution, service levels, and the level of market search. Following this, they focus on the manufacturer’s optimal incentives under two schemes: wholesale prices and holding cost subsidies. They show that the manufacturer might prefer competing retailers when the market search is high. Bartholdi and Kemahlioglu-Ziya (2005) consider two retailers who share a common supplier, in a system with certain service level requirements, in which the supplier keeps separate stock for each retailer and bears all of the inventory risk. They investigate the cooperative game, in which the players can form inventory-pooling coalitions instead of keeping separate stocks and hence increase total profit. They show that mechanisms based on the Shapley value coordinate the supply chain, and they investigate the effects of demand variance and service level requirements on profit allocation. Finally, they focus on the retailers’ collusion against the supplier, which, contrary to intuition, turns out to be not always profitable for the retailers.

Coordination of supply chains through contracts is well studied in the supply chain contracting literature. We refer to Cachon (2003) for a detailed review. Unlike our work, this literature mainly focuses on a single manufacturer and a single retailer, without forecast updates. Two papers that consider forecast updates are Tsay (1999) and Donohue (2000). Tsay (1999) studies quantity flexibility contracts in a two-echelon manufacturer-retailer system with forecast update. He shows, among other results, that quantity flexibility contracts can only coordinate the system if the demand forecasts have zero standard deviation. Donohue (2000) examines a buy-back type contracting scheme between a manufacturer and a retailer in a model with two production modes. Here, the retailer buys products in the first production period and the manufacturer offers a second buying option to the retailer (albeit at a higher price) after the retailer updates his demand forecast. Furthermore, the manufacturer compensates the retailer for unsold products at the end of the selling period at a
buy-back price. Donohue shows that such a contracting scheme can coordinate the system. We re-
mark that this buy-back option differs from ours in that we consider buy-backs after demand forecast
update and not after demand realization.

Finally, we review a paper on information sharing games. Slikker et al. (2003) consider an
information-sharing situation similar to the one analyzed in this paper. However, to increase their
individual expected revenues, the players in their cooperative games can only cooperate through
information sharing, whereas in our games the retailers also benefit from inventory pooling and
coordinated ordering. Slikker et al. show that their class of information games coincides with the
class of cooperative games with a population-monotonic allocation scheme.

3 Preliminaries on cooperative game theory

In this section, we briefly review some concepts from cooperative game theory that are relevant to
our work. Let $N = \{1, ..., n\}$ be a finite set of players. A subset of $S \subseteq N$ is called a coalition, and
$S = N$ is called the grand coalition. A function $v : 2^N \rightarrow \mathbb{R}$, with $v(\emptyset) = 0$, is called a characteristic
function. The value $v(S)$ is interpreted as the maximum total profit that coalition $S$ can obtain
through cooperation without participation of the players outside the coalition. Assuming that the
payoff generated by coalition $S$ can be transferred among the members of $S$, a pair $(N, v)$ is called
a cooperative game with transferable utility (TU game). For a game $(N, v)$, $S \subset N$ and $S \neq \emptyset$, the
subgame $(S, v|_S)$ is defined by $v|_S(T) = v(T)$ for each coalition $T \subseteq S$.

In reality, the players are not interested in benefits of a coalition as much as in their individual
payoffs that arise out of forming that coalition. An allocation is a payoff vector $y = (y_i)_{i \in N} \in \mathbb{R}^N$, specifying for each player $i \in N$ his payoff $y_i$. An allocation $y$ is called efficient if $\sum_{i \in N} y_i = v(N)$, and individually rational if $y_i \geq v(\{i\})$ for all $i \in N$. In an efficient allocation which is individually rational, every player receives in the grand coalition at least as much as he could obtain by staying alone. The set of all individually rational and efficient allocations constitutes the imputation set

$$I(v) = \left\{ y \in \mathbb{R}^N \left| \sum_{i \in N} y_i = v(N) \text{ and } y_i \geq v(\{i\}) \text{ for each } i \in N \right. \right\}.$$

If individual rationality is extended to all coalitions, we obtain the core:

$$\text{Core}(v) = \left\{ y \in \mathbb{R}^N \left| \sum_{i \in N} y_i = v(N) \text{ and } \sum_{i \in S} y_i \geq v(S) \text{ for each } S \subseteq N \right. \right\}.$$

Thus, the core consists of all imputations such that no group of retailers has an incentive to leave
the grand coalition $N$ and form a smaller coalition because they collectively receive at least as much
as they could obtain by themselves.
Bondareva (1963) and Shapley (1967) independently characterized games with a non-empty core by using the notion of balancedness. For \( S \subseteq N \), let us define the vector \( e^S \) as follows:

\[
e^S_i = \begin{cases} 
1, & \text{if } i \in S, \\
0, & \text{otherwise}.
\end{cases}
\]

A map \( \kappa : 2^N/\{\emptyset\} \to [0,1] \) is called a balanced map if \( \sum_{S \in 2^N/\{\emptyset\}} \kappa(S)e^S = e^N \). Further, a game \((N, v)\) is called balanced if \( \sum_{S \in 2^N/\{\emptyset\}} \kappa(S)v(S) \leq v(N) \) for every balanced map \( \kappa : 2^N/\{\emptyset\} \to [0,1] \).

The following theorem is based on Bondareva (1963) and Shapley (1967).

**Theorem 1** Let \((N, v)\) be a TU game. Then \( \text{Core}(v) \neq \emptyset \) if and only if \((N, v)\) is balanced.

A coalitional game \((N, v)\) is called totally balanced if it is balanced and each of its subgames is balanced as well.

## 4 Model with demand updates

In this section, we first introduce a basic setup for our model, and then analyze the effects of different contracting schemes between the manufacturer and the retailers.

### 4.1 Model description and notation

Consider a distribution system that consists of a manufacturer, a warehouse, and \( n \) retailers of an identical product. Each of the retailers faces a stochastic customer demand and has to determine his stocking quantity, which he orders from the manufacturer, at the beginning of a selling period. When placing their orders, the retailers do not know the exact demand realization, but they know the distribution of the demand. After a certain period of time, the goods are produced and shipped to a common location (e.g., a warehouse, port, crossdocking station, etc.). During that period, the retailers may receive a demand signal that updates their information about demand. In our demand-update scenario, we assume that every retailer receives a separate signal about his own demand. After the retailers receive the ordered quantities, the demands are realized and satisfied as much as possible from inventory on hand. The retailers can also salvage any leftover inventory at their locations. As an extension, we consider another salvage opportunity, in the warehouse, which is available only to the manufacturer. This can be interpreted, for example, as a secondary market served by the manufacturer, with a non-overlapping selling period. The manufacturer might redirect the unsold or not-needed units to this market and gain a positive revenue. This opportunity becomes significant if the manufacturer offers buy-back options to the retailers.
To describe the model, we use a tuple

\[ M = (N, (X^i)_{i \in N}, (Y_i)_{i \in N}, t_w, (t_i)_{i \in N}, (p_i)_{i \in N}, (v_i)_{i \in N}, v_w, c_m), \]  

(1)

where the symbols are interpreted as follows:

- \( N \): Set of retailers, \( N := \{1, \ldots, n\} \)
- \( X^i \): Stochastic demand of retailer \( i \), with \( E[X^i] < \infty \) for every \( i \in N \)
- \( Y_i \): Stochastic demand signal of retailer \( i \)
- \( t_w \): Transportation cost from the manufacturer to the warehouse, not including purchasing cost
- \( t_i \): Transportation cost from the warehouse to retailer \( i \)
- \( p_i \): Selling price at retailer \( i \)
- \( v_i \): Salvage value at retailer \( i \)
- \( v_w \): Salvage value at the warehouse
- \( c_m \): Production cost

Throughout the paper, we assume that \( p_i, t_w, t_i, v_i, v_w \) and \( c_m \) are nonnegative. Moreover, we make the following reasonable assumptions. First, we assume that the market is profitable enough – i.e., \( p_i > c_m + t_w + t_i \) for all \( i \in N \). Second, we assume that the salvage opportunities are not a source of profit and that salvaging at the warehouse is more beneficial than salvaging at the retailers: \( c_m + t_w > v_w \) and \( v_w > v_i + t_i \) for all \( i \in N \).

### 4.1.1 Demand signaling process

In this section, we explain how the signaling process works. Consider a probability space \((\Omega, \mathcal{F}, \mathcal{P})\) with \(|\Omega| < \infty^2\). A state of the world \( \omega \in \Omega \) is considered as the markets’ conditions for the retailers. For each state \( \omega \), the markets’ conditions are given by a vector of random variables \((X^i_\omega)_{i \in N}\) where \( X^i_\omega \) is the random demand of retailer \( i \) in state \( \omega \) with distribution function \( F^i_\omega \). Let \( P(\omega) \) denote the probability of having state \( \omega \). Hence, \( \mathcal{P}(E) = \sum_{\omega \in E} P(\omega) \) for all \( E \in \mathcal{F} \). We assume that every retailer \( i \) knows \( P \) and \( X^i_\omega \) for all \( \omega \in \Omega \). Suppose that retailer \( i \) receives updated information and believes that the real state of the world is in a subset \( J_i \) of \( \Omega \). He can update his expectation about the demand as follows: first he deduces posterior probabilities about the states of the world by Bayesian updating – i.e., \( P(\omega|J_i) = P(\omega)/\sum_{j \in J} P(j) \) for every \( \omega \in J_i \). Then, he updates his expectation about the demand to \( X^i_{J_i} \), which distribution is denoted by \( F^i_{J_i} \) and given by \( F^i_{J_i}(x) = \sum_{\omega \in \Omega} P(\omega|J_i) F^i_\omega(x) \) for all \( x \in \mathbb{R} \). Prior to receiving any updated information about the

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\(^2\)We remark that our assumption about finiteness of \( \Omega \) is only for clarity of exposition. All results in this paper would be true for \( \Omega \) being infinite under some mild assumptions on measurability of the functions depending on the random signals.
real state of the world, retailer \(i\) expects a random demand \(X^i\) with distribution function \(F_i\), which is given by \(F^i(x) = \sum_{\omega \in \Omega} P(\omega) P_i^x(\omega)\) for all \(x \in \mathbb{R}\).

To model the knowledge of the retailers after receiving demand signals, we utilize the information partitions by Aumann (1999) in a framework similar to the one used by Slikker (2003). The information partition of retailer \(i\) is a partition of \(\Omega\) and it is denoted by \(I_i\). A partition element is the set of states of the world that the retailer cannot distinguish from each other. For each retailer \(i \in N\), his random signal is a stochastic map \(Y_i : \Omega \rightarrow I_i\). We denote the vectors of random signals by \(Y = (Y_i)_{i \in N}\) and a realization of \(Y\) by \(y = (y_i)_{i \in N}\). We remark that in this information update setting, every random signal of each retailer is consistent – i.e., \(\omega \in Y_i(\omega)\) for all \(\omega \in \Omega\) and all \(i \in N\).

If retailer \(i\) receives a demand signal \(y_i \in I_i\), he learns that the state of the world is in \(y_i\) and expects a random demand \(X^i_{y_i}\), which is derived as above. A special case occurs when the demand signals of the retailers reveal exact demand realizations – i.e., for all \(\omega \in \Omega\) and \(i \in N\), \(X^i_\omega\) is deterministic (i.e., \(X^i_\omega \in \mathbb{R}\)) and \(Y_i(\omega) = \{\bar{\omega} \in \Omega | X^i_\bar{\omega} = X^i_\omega\}\). We call such signals perfect random signals. In the remainder of the paper, we will treat perfect random signals as real numbers and not partitions – i.e., \(Y_i(\omega) = X^i_\omega \in \mathbb{R}\).

In such a situation, in which each retailer receives its own information update, the retailers might improve their knowledge about their future demand if they reveal their signals to each other. Consider a group of retailers \(S \subseteq N\). Suppose that the realization of demand signal appears to be \(y = (y_i)_{i \in N}\), and hence the retailers in \(S\) receive a vector of demand signals \(y^S = (y_i)_{i \in S}\). If they reveal their signals to each other, every retailer \(i \in S\) can update his knowledge \(y_i\) to \(y_S = \cap_{i \in S} y_i\), which further updates their expectation about demand to \(X^i_{y_S}\). The joint information partition of coalition \(S\) is given by

\[
I_S = \{\cap_{i \in S} y_i | y_i \in I_i \text{ for all } i \in S \text{ and } \cap_{i \in S} y_i \neq \emptyset\}.
\]

However, sometimes the retailers cannot further update their knowledge by sharing their demand signals. This happens if the demand signal of a retailer does not reveal any extra information about the demand of the other retailers – i.e., if for all \(y^S = (y_j)_{j \in S} \in \prod_{j \in S} I_j\) and all \(i \in S\), \(X^i_{y_S} = X^i_{y_i}\).

One example is that the retailers receive a public signal \(Y^P\), i.e., \(Y_i = Y^P\) for all \(i \in S\) or \(I_i = I_j\) for all \(i, j \in S\). Another example would be that the retailers serve in completely separate markets and they cannot distinguish between the status of the other retailers’ markets – i.e., \(\Omega = \prod_{i \in N} \Omega_i\), and for all \(i \in N\), for any demand signal \(y_i\) and for all \(\omega_i \in \Omega_i\), all \(\omega\) with \(i^{th}\) coordinate being \(\omega_i\) are either all in \(y_i\) or none of them are in \(y_i\). In the remainder of the paper, if not indicated otherwise,

\(^3\)Note that for every realization \(y\) of \(Y\), each retailer \(i\) observes his associated signal \(y_i\).
we assume that the retailers either do not share their demand signals upon forming a coalition, or their signals do not reveal extra information about others’ demands.

4.2 Centralized system

In this section we consider, as a benchmark, a centralized system in which we assume that a central decision maker (CDM) coordinates the supply chain and determines optimal actions that maximize the total system profit. Under the supervision of a CDM, the total system might benefit if the ordered units are reallocated at the warehouse according to the updated demand information\(^4\). We assume that a CDM makes joint stocking decisions with this assumption in mind.

Let \( q \in \mathbb{R}^+ \) be a joint order quantity for all the retailers. The total expected profit function is

\[
\pi^*(q) = -(c_m + t_w)q + E_Y[G^*(q, \cdot)]
\]

where we use asterisk to denote the expected profit in the centralized system. \( G^*(q, y) \) denotes the total revenue if the joint order, \( q \), is allocated optimally according to demand signal \( y \). If we denote by \( A_i^N \) the share of \( q \) allocated to retailer \( i \) in the centralized model and let \( A^N = (A_i^N)_{i \in N} \), \( G^*(q, y) \) can be expressed as

\[
G^*(q, y) = \max_{A^N \in \mathbb{R}^N_+} H^*(q, y, A^N)
\]

\[
\text{s.t. } \sum_{i \in N} A_i^N \leq q,
\]

where

\[
H^*(q, y, A^N) = -\sum_{i \in N} A_i^N(t_i - v_i) + v_w(q - \sum_{i \in N} A_i^N) + \sum_{i \in N}(p_i - v_i) \int_0^{A_i^N} \tilde{F}_y(x)dx,
\]

and \( \tilde{F}(x) = 1 - F(x) \). Note that we assume that some items may be left in the warehouse when expected demand is low, so that the system benefits from the salvage opportunity there and it does not incur extra transportation cost. Our main result for the centralized system is given in the following theorem.

**Theorem 2** \( \pi^* \) is a concave function, and there exists a finite optimal order quantity, \( q^* \), that maximizes \( \pi^* \).

We refer to the outcome in Theorem 2 as a first-best solution.

\(^4\)We assume that the retailers do not reveal their demand signals to the CDM but their demand forecasts \( F_{y_i} \). CDM only uses the information that is provided to him to make the optimal decision for the system.
4.3 Decentralized system with wholesale price contracts

In this section, we consider a decentralized system in which the manufacturer charges a wholesale price \( w \) for each product ordered by the retailers. To prevent unrealistic cases, we assume that \( w \geq c_m \). We focus on two cases. In the first case, the retailers do not cooperate, and thus they cannot benefit from the updated demand signal. In the second case, the retailers cooperate by reallocating their orders optimally after observing the demand signal, and they coordinate their orders from the manufacturer accordingly.

4.3.1 Non-cooperating retailers

We start by assuming that the retailers are working individually. Because they cannot change their order quantities or take any recourse action after acquiring additional information, they cannot benefit from the updated demand signal. Consequently, if retailer \( i \) orders \( q_i \) units, his profit function is a simple newsvendor-type profit function given by

\[
\pi_i^{\text{NC}}(q_i) = -(w + t_w + t_i - v_i)q_i + (p_i - v_i) \int_0^{q_i} F_i(x)dx,
\]

where superscript \( \text{NC} \) is used to denote non-cooperating retailers. It is well known that for any \( i \in N \), \( \pi_i^{\text{NC}} \) is a concave function, and there exists a finite optimal order quantity, \( q_i^{\text{NC}}(w) \), that maximizes \( \pi_i^{\text{NC}} \). The associated manufacturer’s profit function is then

\[
\pi_m^{\text{NC}}(w) = (w - c_m) \sum_{i \in N} q_i^{\text{NC}}(w).
\]

4.3.2 Cooperating retailers

After observing demand signals, the retailers may increase their total profit if they are able to reallocate the ordered quantities and coordinate their orders from the manufacturer\(^5\). In this subsection, we assume that the retailers may form coalitions to benefit from this feature. First, we analyze the resulting cooperative game and show that it has a non-empty core. Hence, there exists a stable allocation of the joint profit among the members of the grand coalition. After this, we analyze how the manufacturer might be affected by this cooperation.

\(^5\)We remark that if the retailers cooperate, they communicate only their demand forecasts \( F_{y_i} \) and not their demand signals. Since the retailers do not know the forecast update processes of each other (i.e., they do not know the information partitions and \( X_{i,j}'s \) of each other), we do not expect that they can further update their demand information knowing the demand forecasts of the others.
Let $S \subseteq N$ be an arbitrary coalition, and let $q \in \mathbb{R}_+$ be an order placed jointly by all coalition members. Then, the expected profit of coalition $S$ is given by

$$\pi^C_S(q) = -(w + t_w)q + E_Y[G^C_S(q, \cdot)],$$

where superscript $C$ denotes cooperating retailers. $G^C_S(q, y)$ is the total revenue if the joint order is allocated among the retailers optimally, according to demand signal $y$. If we denote by $A_i$ the share of $q$ allocated to retailer $i \in S$ and let $A = (A_i)_{i \in S}$, then $G^C_S(q, y)$ can be expressed as

$$G^C_S(q, y) = \max_{A \in \mathbb{R}^S_+} H^C_S(q, y, A) \quad \text{s.t.} \quad \sum_{i \in S} A_i = q,$$

where

$$H^C_S(q, y, A) = -\sum_{i \in S} A_i(t_i - v_i) + \sum_{i \in S} (p_i - v_i) \int_0^{A_i} F_{y_i}(x) dx.$$

Note that in this case we assume that the manufacturer does not offer any buy-backs and that the retailers have already paid for their orders; therefore the entire quantity ordered, $q$, is distributed among the retailers in $S$ – that is, (4) is satisfied as an equality. As the following theorem can be proven easily by following an approach similar to that used in the proof of Theorem 2, we state it without the proof.

**Theorem 3** $\pi^C_S$ is a concave function, and there exists a finite optimal order quantity, $q^C_S(w)$, that maximizes $\pi^C_S$.

The associated game in coalitional form is a pair $(N, v^C)$ in which the characteristic function $v^C$ assigns to a coalition the maximum total profit that a coalition can obtain – i.e.,

$$v^C(S) = \max_{q \in \mathbb{R}_+} \pi^C_S(q) \quad \text{for all} \ S \subseteq N.$$ 

The following theorem states that there exists a stable allocation of the joint profit among the members of the grand coalition.

**Theorem 4** The game $(N, v^C)$ has a non-empty core.

As mentioned earlier, non-emptiness of the core implies that the joint profit generated by all of the retailers may be allocated among them so that no subset of players is better off on their own. Note that we do not select a specific allocation rule. It is, in general, NP-hard to show whether an
allocation belongs to the core. Chen and Zhang (2006) show that the problem is NP-hard even for a simple collaborative newsvendor game. However, using an approach based on that developed in Chen and Zhang, we can construct a core allocation for this game (see Appendix B).

Thus, the retailers always prefer the model with cooperation to the one without cooperation. It is not obvious how their cooperation impacts the manufacturer. Suppose that the retailers form the grand coalition and order the optimal order quantity, \( q_{N}^{C}(w) \), from the manufacturer. The associated manufacturer’s profit is given by

\[
\pi_{m}^{C}(w) = (w - c_{m})q_{N}^{C}(w).
\]

In both of the cases mentioned so far, the manufacturer’s profit is a function of the total retailers’ order quantity. In the inventory literature, it is known that inventory centralization leads to a decrease in total inventory (order quantity in a newsvendor setting) for many realistic situations. Hence, one would expect that the manufacturer’s profit decreases if the retailers cooperate and order less due to inventory centralization. However, the effect of inventory centralization can actually be the reverse, and that the manufacturer might actually benefit from it. Yang and Schrage (2002) derived sufficiency conditions on demand distributions and cost parameters under which inventory centralization results in higher inventories (higher order quantities in our case). For instance, when symmetric retailers face normally distributed demand, and the overstocking cost is high relative to the understocking cost, their total inventory order increases as a result of collaboration. High overstocking cost discourages individual retailers from ordering larger quantities; however, once they can reallocate potential excess inventory to their partners, they are encouraged to increase their orders. On the other hand, when the understocking cost is high, individual retailers order larger amounts to avoid possible stockouts, while the retailers who collaborate reduce their orders, as potential stockouts may be fulfilled by using inventory initially ordered by another retailer. As a result, manufacturer’s profit increases when the overstocking cost is high, and it decreases when the stockouts are more costly. Yang and Schrage (2002) also show that the inventory levels increase as a result of the retailers’ cooperation for some right-skewed distribution (e.g., exponential, Poisson).

We summarize the above discussion in the following observation.

**Observation 1** A manufacturer’s profit may either increase or decrease as a result of coalition formation. He is more likely to benefit from cooperation when the newsvendor ratio is low, or when demand follows right-skewed distribution.
4.4 Decentralized system with buy-back contracts

In the previous section, we assumed that the retailers have to reallocate the entire quantity that they ordered from the manufacturer. In this section, we assume that some of the inventory may be left at the warehouse. Namely, we consider a decentralized system in which the manufacturer offers a buy-back contract to the retailers. He charges a wholesale price \( w \) for each product ordered by the retailers and promises to buy back extra units at a price \( b \) after the demand information is updated with the demand signal. To prevent unrealistic cases, we assume that \( w \geq b \). Note that this model differs from the traditional buy-back contract, in which the return of inventories occurs only after all uncertainty about demand has been resolved. In this model, some inventory may be returned although the true demand has not yet been revealed. As in the previous section, we first analyze the case with non-cooperating retailers, and then we consider the cooperation among the retailers. Finally, we compare buy-back contracts with wholesale price contracts from the retailers’ and the manufacturer’s points of view.

4.4.1 Non-cooperating retailers

With a buy-back contract, the retailers have the opportunity to return extra items (albeit at a lower price) if, by the time the orders arrive to the warehouse, the demand signal reveals a demand expectation lower than originally predicted. Hence, the expected profit function of retailer \( i \) is

\[
\pi_{BN}^i(q) = -(w + t_w)q + E_Y [G_{BN}^i(q, \cdot)],
\]

where superscript \( BN \) denotes the model with buy-backs and non-cooperating retailers. \( G_{BN}^i(q, y) \) is the expected revenue if retailer \( i \) determines an optimal quantity to return to the manufacturer, given demand signal \( y \). Thus, if \( A \) denotes the quantity actually received by retailer \( i \), then

\[
G_{BN}^i(q, y) = \max_{A \in \mathbb{R}_+} H_{BN}^i(q, y, A)
\]

s.t. \( A \leq q \),

where

\[
H_{BN}^i(q, y, A) = -A(t_i - v_i) + b(q - A) + (p_i - v_i) \int_0^A \bar{F}_y(x)dx.
\]

Following an approach similar to that used in the proof of Theorem 2, the following theorem can be proven easily, so we state it without the proof.
Theorem 5  $\pi_i^{BN}$ is a concave function, and there exists a finite optimal order quantity, $q_i^{BN}(w, b)$, that maximizes $\pi_i^{BN}$.

It can be easily shown that $q_i^{BN}(w, b) \geq q_i^{NC}(w)$ for all $i \in N$, and thus a retailer with a buy-back opportunity always increases his order size. Further, let $A_i^{BN}(y)$ be an optimal allocation\footnote{Because we are searching for an optimal allocation in a non-empty compact set, there exists at least one optimal allocation.} that maximizes $H_i^{BN}(q_i^{BN}(w, b), y, \cdot)$ for demand signal $y$. The associated manufacturer’s profit function is then given by

$$
\pi_m^{BN}(w, b) = (w - c_m) \sum_{i \in N} q_i^{BN}(w, b) - (b - v_w) \sum_{i \in N} EY \left[ q_i^{BN}(w, b) - A_i^{BN}(\cdot) \right].
$$

Note that, for $b = v_w$,

$$
\pi_m^{BN}(w, v_w) = (w - c_m) \sum_{i \in N} q_i^{BN}(w, v_w) \geq (w - c_m) \sum_{i \in N} q_i^{NC}(w) = \pi_m^{NC}(w).
$$

Thus, when the retailers do not cooperate, the manufacturer can always pick a buy-back price that assures that the buy-back contract will make him better off than the wholesale price contract.

4.4.2 Cooperating retailers

In this section, we assume that the retailers can form coalitions in order to benefit from coordinated ordering from the manufacturer and subsequent reallocation of the ordered quantities (after the demand signal is observed). We first analyze the associated cooperative game and show that there is a stable allocation of the total profit for the grand coalition. Then we consider the impact of this cooperation on the manufacturer’s profit.

Let $S \subseteq N$ be an arbitrary coalition, and let $q \in \mathbb{R}_+$ be an order placed jointly by all coalition members. Then, the expected profit of coalition $S$ is given by

$$
\pi_S^{BC}(q) = -(w + t_w)q + EY[G_S^{BC}(q, \cdot)],
$$

where superscript $BC$ denotes the model with buy-backs and cooperating retailers. $G_S^{BC}(q, y)$ is the total revenue if the retailers optimally decide on the amount to return to the manufacturer and reallocate the remaining items in an optimal fashion, given demand signal $y$. Thus, if $A_i$ denotes the quantity actually received by retailer $i$, and we let $A = (A_i)_{i \in S}$, then

$$
G_S^{BC}(q, y) = \max_{A \in \mathbb{R}_+^S} H_S^{BC}(q, y, A)
$$

$$
\text{s.t. } \sum_{i \in S} A_i \leq q,
$$
where

\[ H_{S}^{BC}(q, y, A) = -\sum_{i \in S} A_i(t_i - v_i) + b(q - \sum_{i \in S} A_i) + \sum_{i \in S} (p_i - v_i) \int_{0}^{A_i} F_{y_i}(x) dx. \] (5)

Note that, unlike in (4), the total reallocated quantity does not need to correspond to the ordered amount, as the manufacturer will buy back the leftover items. For our explanatory convenience, we sometimes denote the expected profit of coalition \( S \) as a function of \( w \) and \( b \) as well – i.e., as \( \pi_{S}^{BC}(q, w, b) \). Similarly, we sometimes denote the functions \( G_{S}^{BC} \) and \( H_{S}^{BC} \) as \( G_{S}^{BC}(b, q, y) \) and \( H_{S}^{BC}(b, q, y, A) \), respectively. Following an approach similar to the one used in the proof of Theorem 2, the following theorem can be proven easily, so we state it without the proof.

**Theorem 6** \( \pi_{S}^{BC} \) is a concave function, and there exists a finite optimal order quantity, \( q_{S}^{BC}(w, b) \), that maximizes \( \pi_{S}^{BC} \).

It can be easily shown that \( q_{S}^{BC}(w, b) \geq q_{C}^{S}(w) \), and thus a coalition with a buy-back opportunity always increases its order size. Let \((N, v^{BC})\) be the associated game in coalitional form, where the value of coalition \( S \) is given by

\[ v^{BC}(S) = \max_{q \in \mathbb{R}_+} \pi_{S}^{BC}(q). \]

The following theorem states that there is a stable allocation of the total profit among the members of the grand coalition.

**Theorem 7** The game \((N, v^{BC})\) has a non-empty core.

Thus, similarly to the case without buy-backs, we can allocate the total profit generated by all retailers in a manner that discourages defections because no coalition can generate higher profits on its own. A core allocation can be found following an approach similar to that shown in Appendix B.

Thus, with or without buy-backs, the retailers benefit from cooperation. Suppose that the retailers form the grand coalition and order the optimal quantity \( q_{N}^{BC}(w, b) \) from the manufacturer. Let \( A^{BC}(y) \) be an optimal allocation that maximizes \( H_{N}^{BC}(q_{N}^{BC}(w, b), y, \cdot) \) for demand signal \( y \). The associated manufacturer’s profit is given by

\[ \pi_{m}^{BC}(w, b) = (w - c_m)q_{N}^{BC}(w, b) - (b - v_w)E_{Y}[q_{N}^{BC}(w, b) - \sum_{i \in N} A_i^{BC}(\cdot)]. \]

Note that, for \( b = v_w \),

\[ \pi_{m}^{BC}(w, v_w) = (w - c_m)q_{N}^{BC}(w, v_w) \geq (w - c_m)q_{N}^{C}(w) = \pi_{m}^{C}(w). \] (6)
Thus, when the retailers cooperate, the manufacturer can always pick a buy-back price that assures that the buy-back contract will make him better off than the wholesale price contract.

Similar to the cases with wholesale price contracts, the retailers prefer to cooperate and form the grand coalition if they can allocate the total profit using a core solution. Furthermore, this coalition is stable because none of the groups benefits by defecting from the grand coalition. However, the manufacturer might be either better off or worse off as a result of this cooperation, depending upon the demands’ distributions.

4.4.3 Comparison of buy-back and wholesale price contracts

Compared with the wholesale price contracts, buy-back contracts provide certain advantages for both the retailers and the manufacturer. Although the retailers’ profit may decrease if the manufacturer charges a higher wholesale price with the implementation of buy-backs, note that for a given wholesale price \( w \), both the retailers’ optimal order quantities and their expected profits are increasing with the buy-back price \( b \). Therefore, for a given wholesale price, the retailers always prefer a buy-back contract with a high buy-back price to a wholesale price contract.

With a buy-back contract, the manufacturer enjoys higher order quantities. However, his profit is reduced by the amount that he has to pay for the returns at the warehouse, which is an increasing function of order quantities. By offering a buy-back price equal to the salvage value at the warehouse, \( b = v_w \), the manufacturer’s expected profit in the buy-back setting exceeds the expected profit in the wholesale price setting. Therefore, if we assume that the manufacturer can determine the buy-back price in an optimal fashion, he always prefers a buy-back contract to the wholesale price contract.

5 Achieving a first-best solution with buy-back contracts

In our analysis so far, we have assumed that the wholesale price, \( w \), and the buy-back price, \( b \), were given exogenously. In this section, we consider the manufacturer’s coordination effort with cooperating retailers through the use of the buy-back contracts. Clearly, when the retailers cooperate and act as a single retailer, selection of the coordinating contract parameters should be easier than in the case in which the retailers act independently. For our model with cooperating retailers (see Section 4.2), a system-optimal solution can only be achieved if the following two conditions are satisfied: (i) \textit{quantity-optimality} – the optimal order quantity of the centralized system is ordered, and (ii) \textit{allocation-optimality} – the allocation of the ordered units in the decentralized system matches the allocation in the centralized system.
Note that this problem differs from traditional coordination via buy-back contracts with a single retailer, in which only the quantity-optimality condition has to be satisfied. In addition, note that the returns in this model may happen before the true demand realization is known, which makes the allocation-optimality condition more difficult to satisfy. The following theorem describes our results regarding the quantity-optimality condition.

**Theorem 8** In a newsvendor model with multiple retailers, for any \( v_w \leq b < \min_{i \in N} (p_i - t_i) \), there exists a \( w \) such that \( q_{BC}^N(w, b) = q^* \), and \( w \) is increasing with \( b \).

We call contracts \((w, b)\) described in Theorem 8 *quantity-preserving contracts*. As shown in Theorem 8, the quantity-optimality condition can be satisfied by a broad range of buy-back contracts. However, it is not always possible to satisfy the allocation-optimality condition. We separately analyze two situations: full information and partial information revelation after demand signal is observed.

### 5.1 Full information

In the full information model, the demand signal reveals the exact demand realization. Hence, after the orders arrive at the warehouse, the retailers know the exact demand, and they can reallocate the ordered units accordingly. Because the exact demand is known by the time of reallocation, salvage values at the retailers do not play a role, and an optimal order allocation in the centralized system is optimal in the decentralized system with a buy-back price \( v_w \leq b \leq \min_{i \in N} (p_i - t_i) \) as well. For the cases with \( b > \min_{i \in N} (p_i - t_i) \), the allocation of the orders might not match because the retailer whose revenue is lower than the buy-back price prefers returning the items to satisfying his known demand, which decreases the total profit in the decentralized system. We summarize this analysis in the following proposition, which we state without the proof.

**Proposition 1** When demand signal is perfect, the decentralized system with quantity-preserving contracts achieves a first-best solution. When \( b > \min_{i \in N} (p_i - t_i) \), the decentralized system generates less profit than the centralized one.

When the retailers are symmetric, that is, \( p_i - t_i = p_j - t_j \) for all \( i, j \in N \), demands at different retailers have equal importance because they bring equal revenue. Therefore, the system can easily be reduced to a one-retailer setting, in which this single retailer is facing the total demand of all the retailers. For such a case, it has been shown that a system-optimal solution can be achieved by any quantity-preserving buy-back contract \((w, b)\). In addition, any division of total profits among the manufacturer and the retailers can be achieved by an appropriate selection of \( b \).
5.2 Partial information case

In the partial information case, the demand signal does not reveal the exact demand. Instead, it provides the retailers with an updated demand distribution. In this case, reallocation of the orders in a decentralized system with an quantity-preserving buy-back contract \((w, b)\) may not match the allocation in the centralized system. Hence, the total profit of the decentralized system is, in general, lower than that of the centralized system. Recall that Theorem 8 holds for \(b \geq v_w\), and note that it follows from (2) and (5) that, given the same initial orders, allocations in both systems will be the same whenever \(b = v_w\). The following example illustrates how is the loss in profit when \(b \neq v_w\) affected by the quality of the demand signal and by the value of the buy-back price.

Example 5.1 We consider a single retailer, who may represent multiple cooperating retailers when all retailers charge the same prices and face the same costs. Recall that \(c_m, t_w, v_w, p, v, t\) denote the manufacturing cost, the transportation cost to the warehouse, the salvage value at the warehouse, the selling price, the salvage value at the retailer, and the transportation cost from the warehouse to the retailer, respectively. After the orders arrive at the warehouse, the retailer receives demand signal \(y \in I = \{l, h\}\), which reveals that demand is uniformly distributed on \([y, y + \delta y]\). Signal \(y = l\) represents low-demand expectation, while signal \(y = h\) represents high-demand expectation. We denote by \(P(y)\) the probability of receiving demand signal \(y\), and by \(F_y\) the distribution function of the uniform demand on \([y, y + \delta y]\). Moreover, we assume that \(l + \delta l < h\).

First, we focus on the centralized system, and suppose that \(q^*\) is its optimal order quantity\(^7\). After receiving demand signal \(y\), the CDM decides on the optimal quantity, \(A^*_y\), to send to the retailer, which maximizes

\[
H^*_y(A) = (-v_w - t + v)q + (p - v) \int_{0}^{q} F_y(x) dx,
\]

s.t. \(q^* \geq A \geq 0\).

Since \(H^*_y\) is concave, it is easy to derive that \(A^*_y = \min\{q^*, y + \delta y \frac{v_w - t}{p - v}\}\). The total profit of the centralized system is

\[
\pi^*(q^*) = -(c_m + t_w)q^* + \sum_{y \in I} P(y) \left((q^* - A^*_y)v_w - A^*_y(t - v) + (p - v) \int_{0}^{A^*_y} F_y(x) dx\right).
\]

Now, consider a quantity-preserving buy-back contract. Since the contract is quantity-preserving, the retailer orders \(q^*\) units in the decentralized system as well. If the retailer receives signal \(y\), he

\(\text{We remark that } q^* \in [0, h + \delta h \frac{v_w - t}{p - v}].\)
solves the following problem to decide how many units, \( A^R_y \), to keep:

\[
\max \quad H^R_y(A) = (-b - t + v)A + (p - v) \int_0^A \bar{F}_y(x)dx,
\]

\[
\text{s.t.} \quad q^* \geq A \geq 0.
\]

Similarly as above, we can derive that \( A^R_y = \min\{q^*, y + \delta_y\frac{p-b-t}{p-v}\} \). We remark that \( A^R_y \leq A^*_y \) for all \( y \in I \) since \( b \geq v_w \). The total profit of the decentralized system is, therefore,

\[
\pi^{BC}(q^*) = -(c_m + t_w)q^* + \sum_{y \in I} P(y) \left( (q^* - A^R_y)v_w - A^R_y(t - v) + (p - v) \int_0^{A^R_y} \bar{F}_y(x)dx \right).
\]

The difference between total system profits in the centralized and the decentralized system is

\[
\pi^*(q^*) - \pi^{BC}(q^*) = \sum_{y \in I} P(y) \left( (q^* - A^*_y)v_w - A^*_y(t - v) + (p - v) \int_0^{A^*_y} \bar{F}_y(x)dx \right.
\]

\[
-(q^* - A^R_y)v_w + A^R_y(t - v) - (p - v) \int_0^{A^R_y} \bar{F}_y(x)dx
\]

\[
= \sum_{y \in I} P(y) \left( -[A^*_y - A^R_y] (v_w + t - v) + (p - v) \int_{A^R_y}^{A^*_y} \frac{\delta_y + y - x}{\delta_y} dx \right). \tag{7}
\]

Since \( A^R_y = \min\{q^*, y + \delta_y\frac{p-b-t}{p-v}\} \), \( A^*_y = \min\{q^*, y + \delta_y\frac{p-v-w-t}{p-v}\} \) and \( b \geq v_w \) for all \( y \in I \), it can be seen from (7) that the the gap between two profits is increasing with the optimal order quantity. We summarize the results for different optimal order quantities as follows:

<table>
<thead>
<tr>
<th>( q^* )</th>
<th>( \pi^<em>(q^</em>) - \pi^{BC}(q^*) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 \leq q^* \leq l + \delta_l\frac{p-b-t}{p-v} )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( l + \delta_l\frac{p-b-t}{p-v} &lt; q^* \leq l + \delta_l\frac{p-v-w-t}{p-v} )</td>
<td>( P(l) \left( -[A^<em><em>l - A^R_l] (v_w + t - v) + (p - v) \int</em>{A^R_l}^{A^</em>_l} \frac{\delta_l + l - x}{\delta_l} dx \right) )</td>
</tr>
<tr>
<td>( l + \delta_l\frac{p-v-w-t}{p-v} &lt; q^* \leq h + \delta_h\frac{p-b-t}{p-v} )</td>
<td>( P(l) \frac{\delta_l}{p-v} \left( \frac{b-v}{2} \right)^2 )</td>
</tr>
<tr>
<td>( h + \delta_h\frac{p-b-t}{p-v} &lt; q^* \leq h + \delta_h\frac{p-v-w-t}{p-v} )</td>
<td>( P(h) \left( -[A^<em><em>h - A^R_h] (v_w + t - v) + (p - v) \int</em>{A^R_h}^{A^</em>_h} \frac{\delta_h + h - x}{\delta_h} dx \right) )</td>
</tr>
</tbody>
</table>

Thus, as noted above, the profits in the two models coincide only when \( b = v_w \), which results in \( A^*_y = A^R_y \) for all \( y \in I \); for other choices of the buy-back value, the centralized system performs better than the decentralized one. Note that the gap between the two profits can be arbitrarily large,
depending upon the values of $q^*, \delta_l, \delta_h, b - v_w$, and $p - v$. This also shows the value of information for the model with imperfect demand updates.

As can be seen from the example, the profits in the centralized and decentralized system coincide if $b = v_w$, or when no inventory is salvaged at the warehouse. Note that the second case is likely to occur with two-point demand distributions (or their variations, as the one presented in the example) and low critical fractile values. As demand in practice is more likely to exhibit continuous distribution, or discrete distribution with more than two points, it appears that there will exist a profit difference between the two models whenever $b \neq v_w$. Factors such as high uncertainty in the demand signal, low salvage value, or high buy-back price are likely to lead to a worse performance of the decentralized model. Therefore, the manufacturer in a decentralized system may prefer to offer a lower wholesale price (and consequently a lower buy-back price) or to reduce demand uncertainty and thus increase the total system profit. This leads to the following result, which we state without proof.

**Theorem 9** When demand signal is imperfect, the following statements hold.

1. A first-best outcome is achieved whenever $b = v_w$ and a quantity-preserving contract is used.

2. When $b \neq v_w$, returns in the decentralized system do not, in general, match the returns in the centralized system, even when the order quantities in both systems coincide. The gap between the profits in the two systems can be arbitrarily large.

As we have shown in (6), the manufacturer’s profit when he offers buy-back price $b = v_w$ always exceeds his profit with the wholesale price contract. Thus, by using quantity-preserving contract with $b = v_w$, the manufacturer maximizes the system profit and, at the same time, increases his own profit. Note that this does not imply that the manufacturer cannot achieve higher profit with another combination of $w$ and $b$.

6 Cooperation between the retailers with information sharing

In this section, we again assume that the manufacturer offers buy-back contracts to the retailers, but use different assumptions about cooperation among the retailers. Unlike the analysis in the previous sections, here we assume that the demand signals observed by the retailers might reveal extra information about the other retailers’ demands, and that coalition members share this information among themselves. We show that there exists a stable allocation of total profit for the grand coalition.
We first consider the effect of information sharing. Suppose that the realization of demand signals are \( y = (y_i)_{i \in N} \). Let \( S \subseteq N \) be an arbitrary coalition, whose members observe a demand signal vector \( y^S = (y_i)_{i \in S} \). If we assume that coalition members are sharing their information, then every retailer updates his knowledge about the state of the world to \( y_S = \cap \{i \in S \} y_i \). Hence, retailer \( i \) expects a random demand \( X^i_{y_S} \) with distribution function \( F^i_{y_S} \). The effect of information sharing is twofold. First, the exact realization of the profit might be different from what a retailer is expecting without information sharing. This does not occur if the retailers’ demand signals do not provide any extra information about each other’s demand, or if the retailers share their demand information. Second, the retailers can make better ordering and allocation decisions by sharing their information with each other and further updating their demand forecasts. We formalize this discussion in the following theorem, and illustrate the effects of information sharing with an example.

**Theorem 10** For any coalition \( S \subseteq N \), information sharing leads to better ordering and allocation decisions in expectation.

**Example 6.1** Consider a model \( M \) described by (1), wherein \( N = \{1, 2\} \), \( t_i = 0 \), \( p_i = p \), and \( v_i = v \) for all \( i \in N \). Let \( w \) and \( b \) be the wholesale price and buy-back price in the buy-back contract, and let \( b = 0 \). Hence, we can treat the situation as there is a wholesale price contract between the retailers and the manufacturer. Let \( R = \frac{p - w}{p - v} = 0.5 \) be the newsvendor ratio. Assume that \( \Omega \) consists of five elements \( \{(l, l), (h_1, l), (l, h_1), (h_1, h_1), (h_2, h_2)\} \), with probabilities \( \{0.48, 0.12, 0.12, 0.03, 0.25\} \), respectively. Each element in \( \Omega \) represents the exact demand for the retailers. Thus, \( (l, l) \) represents low demand for each retailer, whereas \( (l, h_1) \) means low demand for retailer 1 and high demand for retailer 2. Moreover, states \( h_1 \) and \( h_2 \) represent the same high demand. Assume that the information partitions are such that individual retailers can only distinguish a part of their high demand and also have information about each other’s demand\(^9\). Thus, the information partitions of the retailers and the joint information partition of the coalition are given by

\[
I_1 = \left\{ \{ (l, l), (h_1, l) \}, \{ (l, h_1), (h_1, h_1) \}, \{ (h_2, h_2) \} \right\};
\]

\[
I_2 = \left\{ \{ (l, l), (l, h_1) \}, \{ (h_1, l), (h_1, h_1) \}, \{ (h_2, h_2) \} \right\};
\]

\[
I_{\{1,2\}} = \left\{ \{ (l, l), (h_1, l), (l, h_1), (h_1, h_1), (h_2, h_2) \} \right\}.
\]

\(^8\)We remark that in the case without information sharing retailer \( i \) knows only that the state of the world is in \( y_i \), and he expects a random demand \( X^i_{y_i} \) with distribution function \( F^i_{y_i} \).

\(^9\)We remark that since the retailers do not know each other’s demand forecast update process, they cannot come up with each others demand forecast knowing the state of the world.
First, consider the case without information sharing. For each realization of $\Omega$, the retailers do not have enough information to allocate the joint order optimally and they simply split it equally among themselves. Hence, the coalition acts as if the retailers work individually expecting a low demand with probability 0.6 and a high demand with probability 0.4. The optimal order quantity is equal to $2l$, and the expected profit is given by $\pi = 2l(p - w)$.

Now, consider the information sharing case. If the retailers share their information, they can distinguish all states of the world. Observe that the optimal order quantity and the optimal expected profit depend upon the total demand observed by the retailers. In this case, the coalition acts like a single newsvendor facing $2l$ demand with probability 0.48, $l + h$ demand with probability 0.24, and $2h$ demand with probability 0.28. The optimal order quantity can be calculated as $l + h$, and the expected profit is given by $\bar{\pi} = 0.48(2lp + (h - l)v) + 0.52(h + l)p - (h + l)w$. By using $\frac{p-w}{p-v} = 0.5$, simple algebra gives the profit difference of $\bar{\pi} - \pi = 0.02(h - l)(p - v)$, hence the retailers improve their expected profit by sharing their information. ♦

Next, we want to analyze whether information sharing may benefit all retailers simultaneously, that is, if the core is nonempty. Consider coalition $S \subseteq N$, and let $q \in R_+$ be an order placed jointly by all coalition members. Then, the expected profit of coalition $S$ is given by

$$\pi_{S}^{I S}(q) = -(w + t_w)q + E_Y[G_{S}^{I S}(q, \cdot)],$$

where superscript $I S$ denotes the model with information sharing. $G_{S}^{I S}(q, y)$ is the total revenue generated if the retailers optimally decide on the amount to return to the manufacturer and reallocate the remaining items in an optimal fashion, given demand signal $y$. Thus, if $A_i$ denotes the quantity actually received by retailer $i$, and we let $A = (A_i)_{i \in S}$, then

$$G_{S}^{I S}(q, y) = \max_{A \in R_+^S} H_{S}^{I S}(q, y, A)$$

s.t. $\sum_{i \in S} A_i \leq q$,

where

$$H_{S}^{I S}(q, y, A) = -\sum_{i \in S} A_i(t_i - v_i) + b(q - \sum_{i \in S} A_i) + \sum_{i \in S}(p_i - v_i) \int_0^{A_i} \hat{F}_{yS_i}(x)dx.$$

Note that the retailers who share their information further update their demand forecast to $\hat{F}_{yS}$ (instead of $\hat{F}_{yi}$ without information sharing), and they share these forecasts with their coalition members to determine an optimal allocation of the joint order quantity.
Following an approach similar to the one used in the proof of Theorem 2, the following theorem can be proven easily, so we state it without the proof.

**Theorem 11** \( \pi^{IS}_S \) is a concave function, and there exists a finite optimal order quantity, \( q^{IS}_S(w, b) \), that maximizes \( \pi^{IS}_S \).

Let \( (N, v^{IS}) \) be the associated game in coalitional form, where the value of coalition \( S \) is given by

\[
v^{IS}(S) = \max_{q \in \mathbb{R}^+} \pi^{IS}_S(q).
\]

The following theorem states that there is a stable allocation of the total profit among the members of the grand coalition.

**Theorem 12** The game \( (N, v^{IS}) \) has a non-empty core.

An actual core allocation can be calculated using the approach shown in Appendix B, by considering related game \( (N, w^{BC}) \) as in the proof of Theorem 12.

### 7 Conclusion

In this paper, we have studied a multi-retailer distribution system with updated demand information. We have first focused on the opportunity for cooperation among the retailers. Cooperating retailers can reallocate ordered quantities after a demand update signal is observed, and they can coordinate initial orders accordingly. We have considered two possibilities: wholesale price contracts and buyback contracts between the manufacturer and the retailers. We were able to show the non-emptiness of the core under both contracts, thus extending the results that hold for a simple newsvendor setting with exact demand information. Consequently, the retailers are always able to select an allocation of profits under which they prefer the grand coalition to any other alliance structure. We show how such an allocation can be constructed.

We have next analyzed the relationship between the manufacturer and the retailers. Inevitably, the manufacturer is affected by the retailers’ cooperation under both contracts. However, this impact might be in either a positive or negative direction because centralization might lead to an increase or to a decrease in the total ordering quantity (depending upon the parameters such as, for example, demand distributions and the newsvendor ratio). Thus, a strong manufacturer may encourage retailers’ cooperation for low-demand products or critical products with high overstocking costs, while he may try to prevent this for products that are in high demand or have high understocking...
cost. When we compare the two contracts, it is obvious that the buy-back contract provides more flexibility for the manufacturer to manipulate the retailers’ ordering decisions to his own advantage. With an opportunity to redirect the returned items to a secondary market, the importance of such a contracting scheme increases for the manufacturer as well as for the total distribution system.

Finally, we have analyzed a possibility of achieving a system-optimal solution through the use of buy-back contracts when the retailers cooperate with one another. Unlike the traditional case, in which buy-backs occur after demand realization and a system-optimal solution can be achieved with an appropriate selection of the buy-back price, we show that a system-optimal solution with the updated demand information can only be achieved in some special cases (e.g., when the signal reveals the exact demand realization and the retailers are symmetric). If the signal does not reveal the exact demand, the decentralized system cannot reach the profit level of the centralized system (even if the optimal order quantities are induced by an appropriate combination of the wholesale price and the buy-back value) unless the buy-back value corresponds to the salvage value at the warehouse. Furthermore, the gap between the profits in the centralized and the decentralized model can be arbitrarily large and is increasing with factors such as the uncertainty of the demand and the buy-back price. Consequently, the manufacturer may benefit by offering a lower wholesale price and a lower buy-back price or by reducing the uncertainty in the system, which results in an increase in the total system profit.

References


Amsterdam.


and Economic Behavior 31:26-49.


Appendix A - Proofs

Theorem 2 \( \pi^* \) is a concave function, and there exists a finite optimal order quantity, \( q^* \), that maximizes \( \pi^* \).

**Proof of Theorem 2:** We will first show the concavity. Let \( \bar{H}^* : \mathbb{R}^{N \cup \{n+1\}} \times \prod_{i \in N} I_i \rightarrow \mathbb{R} \) be the function defined by

\[
\bar{H}^*(\tilde{A}, y) = -\sum_{i \in N} \bar{A}_i (t_i - v_i) + v_w \bar{A}_{n+1} + \sum_{i \in N} (p_i - v_i) \int_0^{y_i} F_i(x) dx.
\]

It is easy to check that \( \bar{H}^* \) is jointly concave in \( \tilde{A} \). Moreover, we easily verify that \( G^*(q, y^N) = \max_{A \in \{A' \in \mathbb{R}_+^{N\cup\{n+1\}} \mid \sum_{i \in N\cup\{n+1\}} A'_i = q\}} H^*(\tilde{A}, y^N) \).

Then, from Theorem 5.7 of Rockafellar (1970), \( G^* \) is a concave function of \( q \). Taking expectation over \( Y \), \( E_Y[G^*(q, \cdot)] \) is concave, too. Since \( -(c_m + t_w)q \) is also concave, \( \pi^* \) is concave as a sum of two concave functions.

To prove the existence of a finite optimal order quantity, it is enough to show that any order quantities outside a compact range result in non-positive expected profit. Let

\[
\varrho^* = \sum_{i \in N} (p_i - t_i) E[X_i] / (c_m + t_w - v_w).
\]

Then, for all \( q > \varrho^* \), we have

\[
\pi^*(q) \leq -q \cdot (c_m + t_w - v_w) + \sum_{i \in N} (p_i - v_w) E[X_i] \leq 0.
\]

The first inequality can be verified by taking into consideration a lower bound of the difference between transportation costs and the maximum salvage value in the network, and an upper bound of expected revenues. The second inequality follows from definition of \( \varrho^* \). Hence, the optimal order vector exists and belongs to the compact set \([0, \varrho^*]\).

Theorem 4 The game \((N, v^C)\) has a non-empty core.

**Proof of Theorem 4:** We prove the theorem by showing that the game is balanced. Consider an arbitrary \( y \in Y \). Let \( q_S \) be the optimal order quantity of coalition \( S \), and let \( \mathbf{A}^S \) be an optimal allocation that maximizes \( H^C_S(q_S, y, \mathbf{A}) \), subject to \( \mathbf{A} \in \mathbb{R}_+^S \) and \( \sum_{i \in S} A_i = q_S \). We remark that an
optimal allocation exists since we are searching for an optimal allocation in a non-empty compact set. Let $\kappa$ be a balanced map of $N$. Then

$$
\sum_{S \subseteq N} \kappa(S)G^C_N(\kappa_S, y) = \sum_{S \subseteq N} \kappa(S)H^C_N(q_S, y, A^S)
$$

$$
= \sum_{S \subseteq N} \kappa(S) \left( - \sum_{i \in S} A^S_i (t_i - v_i) + \sum_{i \in S} (p_i - v_i) \int_0^{A^S_i} \bar{F}_y(x) \, dx \right)
$$

$$
= \sum_{i \in N} \sum_{S \subseteq N \mid i \in S} \kappa(S) \left( - A^S_i (t_i - v_i) + (p_i - v_i) \int_0^{A^S_i} \bar{F}_y(x) \, dx \right) \quad (A1a)
$$

$$
\leq \sum_{i \in N} \sum_{S \subseteq N \mid i \in S} \kappa(S)A^S_i (t_i - v_i) + \sum_{i \in N} (p_i - v_i) \int_0^{A^S_i} \bar{F}_y(x) \, dx \quad (A1b)
$$

$$
= H^C_N \left( \sum_{S \subseteq N} \kappa(S)q_S, y, \left( \sum_{S \subseteq N \mid i \in S} \kappa(S)A^S_i \right)_{i \in N} \right) \quad (A1c)
$$

$$
\leq G^C_N \left( \sum_{S \subseteq N} \kappa(S)q_S, y \right) \quad (A1d)
$$

(A1a) follows from equation

$$
\sum_{S \subseteq N} \kappa(S) \sum_{i \in S} A^S_i = \sum_{i \in N} \sum_{S \subseteq N \mid i \in S} \kappa(S)A^S_i.
$$

Note that $\kappa(S) \geq 0$ for all $S \subseteq N$ and $\sum_{S \subseteq N \mid i \in S} \kappa(S) = 1$ for all $i \in N$ because $\kappa$ is a balanced map. (A1b) holds because $-A^S_i (t_i - v_i) + (p_i - v_i) \int_0^{A^S_i} \bar{F}_y(x) \, dx$ is a concave function in $A$. (A1c) holds because

$$
\left( \sum_{S \subseteq N \mid i \in S} \kappa(S)A^S_i \right)_{i \in N}
$$
is a feasible allocation of $\sum_{S \subseteq N} \kappa(S)q_S$ - i.e., $\sum_{i \in N} \sum_{S \subseteq N \mid i \in S} \kappa(S)A^S_i = \sum_{S \subseteq N} \kappa(S)q_S$.

(A1d) holds because $G^C_N$ considers an optimal allocation. Now,

$$
\sum_{S \subseteq N} \kappa(S)v^C(S) = \sum_{S \subseteq N} \kappa(S)\pi^C_S(q_S) = - \sum_{S \subseteq N} \kappa(S)q_S(w + t_w) + E_Y \left[ \sum_{S \subseteq N} \kappa(S)G^C_N(q_S, \cdot) \right]
$$

$$
\leq - \sum_{S \subseteq N} \kappa(S)q_S(w + t_w) + E_Y \left[ \sum_{S \subseteq N} \kappa(S)G^C_N(q_S, \cdot) \right] = \pi^C_N \left( \sum_{S \subseteq N} \kappa(S)q_S \right)
$$

$$
\leq v^C(N).
$$

First inequality holds by (A1). This completes the proof.
Theorem 7  The game $(N, v^{BC})$ has a non-empty core.

Proof of Theorem 7: We prove the theorem by showing that the game is balanced. Consider a $y \in Y$. Let $q_S$ be the optimal order quantity of coalition $S$, and let $A^S$ be an optimal allocation that maximizes $H^\text{BC}_S(q_S, y, A)$ subject to $A \in \mathbb{R}^S_+$ and $\sum_i A_i \leq q_S$. Let $\kappa$ be a balanced map of $N$. Then

\[
\sum_{S \subseteq N} \kappa(S) G^\text{BC}_S(q_S, y) = \sum_{S \subseteq N} \kappa(S) H^\text{BC}_S(q_S, y, A^S)
\]

\[
= \sum_{S \subseteq N} \kappa(S) \left( -\sum_{i \in S} A_i^S(t_i - v_i) + b(q_S - \sum_{i \in S} A_i^S) + \sum_{i \in S} (p_i - v_i) \int_0^{A_i^S} \hat{F}_{y_i}(x) \, dx \right)
\]

\[
= \sum_{i \in N} \sum_{S \subseteq N \mid i \in S} \kappa(S) \left( -A_i^S(t_i - v_i) - bA_i^S + (p_i - v_i) \int_0^{A_i^S} \hat{F}_{y_i}(x) \, dx \right) + \sum_{S \subseteq N} \kappa(S)bq_S
\]

\[
\leq -\sum_{i \in N} \sum_{S \subseteq N \mid i \in S} \kappa(S)A_i^S(t_i - v_i) + b \left( \sum_{S \subseteq N} \kappa(S)q_S - \sum_{i \in N} \sum_{S \subseteq N \mid i \in S} \kappa(S)A_i^S \right)
\]

\[
+ \sum_{i \in N} (p_i - v_i) \int_0^{A_i^S} \hat{F}_{y_i}(x) \, dx
\]

\[
= H^\text{BC}_N \left( \sum_{S \subseteq N} \kappa(S)q_S, y, \left( \sum_{S \subseteq N \mid i \in S} \kappa(S)A_i^S \right)_{i \in N} \right)
\]

\[
\leq G^\text{BC}_N \left( \sum_{S \subseteq N} \kappa(S)q_S, y \right).
\]

(A2a) follows from

\[
\sum_{S \subseteq N} \kappa(S) \sum_{i \in S} A_i^S = \sum_{i \in N} \sum_{S \subseteq N \mid i \in S} \kappa(S)A_i^S.
\]

Note that $\kappa(S) \geq 0$ for all $S \subseteq N$ and $\sum_{S \subseteq N \mid i \in S} \kappa(S) = 1$ for all $i \in N$ because $\kappa$ is a balanced map.

(A2a) holds because $-A(t_i - v_i) - bA + (p_i - v_i) \int_0^{A_i^S} \hat{F}_{y_i}(x) \, dx$ is a concave function in $A$. (A2b) holds because $\left( \sum_{S \subseteq N \mid i \in S} \kappa(S)A_i^S \right)_{i \in N}$ is a feasible allocation of $\sum_{S \subseteq N} \kappa(S)q_S$ – i.e., $\sum_{i \in N} \sum_{S \subseteq N \mid i \in S} \kappa(S)A_i^S \leq \sum_{S \subseteq N} \kappa(S)q_S$.
Lemma 1: The optimal allocation is independent of \( w \);

2. For \( b < b^* \), \( \mathbf{A}^N(b, q, y) \geq \mathbf{A}^N(b^*, q, y) \);

3. For \( q < q^* \), \( \mathbf{A}^N(b, q^*, y) \geq \mathbf{A}^N(b, q, y) \);

First inequality holds by (A2). This completes the proof. \( \square \)

**Theorem 8** In a newsvendor model with multiple retailers, for any \( v_w \leq b < \min_{i\in N}(p_i - t_i) \), there exists a \( w \) such that \( q^*_N(b, w) = q^* \), and \( w \) is increasing with \( b \).

**Proof of Theorem 8:** In the proof, we will treat \( q^*_N(b, w) \) as

\[
q^*_N(b, w) = \arg \max_{q \in \mathbb{R}_+} \{ \pi^*_N(q, w, b) \},
\]

whereas, in the main text, \( q^*_N(b, w) \) denotes an optimal order quantity. Note that \( q^*_N(b, w) \) can be a multi-valued function. With this definition, we can restate the theorem as follows: in a model with multiple retailers, for any buy-back price \( b \) with \( v_w \leq b < \min_{i\in N}(p_i - t_i) \), there exists a \( w \) such that \( q^* \in q^*_N(b, w) \), and \( w \) is increasing with \( b \). We first show several properties of optimal allocations and of \( q^*_N(b, w) \). We afterwards use those properties to prove the arguments in the theorem.

For given \( q, w, b \), and demand signal \( y \), the optimal allocation of the grand coalition is given by

\[
\mathbf{A}^N(b, q, y) = \arg \max_{\mathbf{A} \in \mathbb{R}_+^N} \left\{ -\sum_{i \in N} A_i(t_i - v_i) + b(q - \sum_{i \in N} A_i) + \sum_{i \in N} (p_i - v_i) \int_{0}^{A_i} \hat{F}_{y_i}(x)dx \right\}
\]

\[
= \arg \max_{\mathbf{A} \in \mathbb{R}_+^N} \left\{ -\sum_{i \in N} A_i(t_i - v_i) - b \sum_{i \in N} A_i + \sum_{i \in N} (p_i - v_i) \int_{0}^{A_i} \hat{F}_{y_i}(x)dx \right\}.
\]

Since the optimal allocations are determined myopically (according to decreasing marginal contributions), their following properties, which we use in the subsequent analysis, can be checked easily.

**Lemma 1**

1. The optimal allocation is independent of \( w \);

2. For \( b < b^* \), \( \mathbf{A}^N(b, q, y) \geq \mathbf{A}^N(b^*, q, y) \);

3. For \( q < q^* \), \( \mathbf{A}^N(b, q^*, y) \geq \mathbf{A}^N(b, q, y) \);

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Lemma 3

\[ q^* - q \geq \sum_{i \in N} A_i^N(b, q^*, y) - \sum_{i \in N} A_i^N(b, q, y) \geq 0. \]

Proof of Lemma 3:
The inequality holds since

\[ \pi^R_C N b, q, y ] - E_Y [G^R_C N b, q, \cdot] \]
\[ \geq -(w^* + t_w)(q - \tilde{q}) + E_Y [G^R_C N b, q, \cdot] - E_Y [G^R_C N b, \tilde{q}, \cdot] \]
\[ = \pi^R_C N q, w^*, b - \pi^R_C N \tilde{q}, w^*, b. \]

The inequality holds since \( q - \tilde{q} \geq 0 \) and \( w^* > w \).

Lemma 2

For any \( b, q^R_C N b, w \) is monotone decreasing$^{10}$ in \( w \).

Proof of Lemma 2:
As \( \pi^R_C N q \) is concave in \( q \) (from Theorem 6), it is enough to check that \( \pi^R_C N q, w, b - \pi^R_C N \tilde{q}, w, b \) is decreasing in \( w \) for all \( q \geq \tilde{q} \geq 0 \). Let \( w < w^* \); then,

\[
\pi^R_C N q, w, b - \pi^R_C N \tilde{q}, w, b = -(w + t_w)(q - \tilde{q}) + E_Y [G^R_C N b, q, \cdot] - E_Y [G^R_C N b, \tilde{q}, \cdot] \geq -(w^* + t_w)(q - \tilde{q}) + E_Y [G^R_C N b, q, \cdot] - E_Y [G^R_C N b, \tilde{q}, \cdot] = \pi^R_C N q, w^*, b - \pi^R_C N \tilde{q}, w^*, b.
\]

The inequality holds since \( q - \tilde{q} \geq 0 \) and \( w^* > w \).

Lemma 3

For any \( w, q^R_C N b, w \) is monotone increasing in \( b \).

Proof of Lemma 3:
Similarly as in Lemma 2, it suffices to show that \( \pi^R_C N q, w, b - \pi^R_C N \tilde{q}, w, b \) is increasing in \( b \) for all \( q \geq \tilde{q} \geq 0 \). As \( \pi^R_C N b, q, y \) depends on \( b \) only through \( G^R_C N b, q, y \), we need to show that \( G^R_C N b, q, y \) is increasing in \( b \) for all \( q \geq \tilde{q} \geq 0 \). Let \( y \) be a realization of \( Y \), and let \( \Delta = A^N(b, \tilde{q}, y) - A^N(\bar{b}, \tilde{q}, y) \leq 0 \) with \( \bar{b} \geq b \). Then,

\[
G^R_C N b, q, y - G^R_C N \bar{b}, \tilde{q}, y = H^R_C N (\bar{b}, q, y, A^N(b, q, y)) - H^R_C N (b, \tilde{q}, y, A^N(b, \tilde{q}, y)) \]
\[
= H^R_C N (\bar{b}, q, y, A^N(b, q, y) + \Delta) - (\bar{b} - b) \left( q - \sum_{i \in N} A_i^N(b, q, y) \right) \\
+ \bar{b} \sum_{i \in N} \Delta_i + \sum_{i \in N} (p_i - v_i) \int F^i_{y_1}(x)dx - H^R_C N (\bar{b}, \tilde{q}, y, A^N(b, \tilde{q}, y) + \Delta) \\
+ (\bar{b} - b) \left( \tilde{q} - \sum_{i \in N} A_i^N(b, \tilde{q}, y) \right) - \bar{b} \sum_{i \in N} \Delta_i - \sum_{i \in N} (p_i - v_i) \int F^i_{y_1}(x)dx.
\]

10If \( q^R_C N b, w \) is an interval valued function, we use the following definition. We call interval valued function \( f : R \rightarrow R \) monotone increasing if for all \( x < y \) \( \sup f(x) \leq \sup f(y) \) and \( \inf f(x) \leq \inf f(y) \). \( f \) is called monotone decreasing if \( -f \) is monotone increasing.
\[
H_N^{BC}(\bar{b}, q, y, A^N(b, q, y) + \triangle) - H_N^{BC}(\bar{b}, q, y, A^N(b, \bar{q}, y) + \triangle) \\
-(\bar{b} - b) \left[ \left( q - \sum_{i \in N} A_i^N(b, q, y) \right) - \left( \bar{q} - \sum_{i \in N} A_i^N(b, \bar{q}, y) \right) \right] \\
+ \sum_{i \in N} (p_i - v_i) \left( \begin{array}{cc}
A_i^N(b, q, y) & A_i^N(b, \bar{q}, y) \\
A_i^N(b, q, y) & A_i^N(b, \bar{q}, y) + \triangle_i
\end{array} \right) \\
\leq H_N^{BC}(\bar{b}, q, y, A^N(b, q, y) + \triangle) - H_N^{BC}(\bar{b}, q, y, A^N(b, \bar{q}, y) + \triangle) \quad \text{(A4a)} \\
\leq H_N^{BC}(\bar{b}, q, y, A^N(b, q, y)) - H_N^{BC}(\bar{b}, q, y, A^N(b, \bar{q}, y)) \quad \text{(A4b)} \\
= c_N^{BC}(\bar{b}, q, y) - c_N^{BC}(\bar{b}, \bar{q}, y).
\]

Here, (A4a) holds because

\[
-(\bar{b} - b) \left[ \left( q - \sum_{i \in N} A_i^N(b, q, y) \right) - \left( \bar{q} - \sum_{i \in N} A_i^N(b, \bar{q}, y) \right) \right] \\
+ \sum_{i \in N} (p_i - v_i) \left( \begin{array}{cc}
A_i^N(b, q, y) & A_i^N(b, \bar{q}, y) \\
A_i^N(b, q, y) & A_i^N(b, \bar{q}, y) + \triangle_i
\end{array} \right)
\]

is non-positive, since \( q - \sum_{i \in N} A_i^N(b, q, y) \geq \bar{q} - \sum_{i \in N} A_i^N(b, \bar{q}, y) \) and \( A^N(b, q, y) \geq A^N(b, \bar{q}, y) \) from items 3 and 4 in Lemma 1, and \( \bar{F}_{yi}(x) \) is decreasing for all \( i \in N \). (A4b) holds from the definition of \( \triangle \) and because \( A^N(b, q, y) \) maximizes \( H_N^{BC}(\bar{b}, q, y, A) \) with \( A \in \mathbb{R}^N \) and \( 0 \leq \sum_{i \in N} A_i \leq q \). \( \square \)

**Lemma 4** Let \( b \in [v_w, \min_{i \in N}(p_i - t_i)] \). Then, there exist a \( q \in q_N^{BC}(c_m, b) \) such that \( q \geq q^* \). Moreover, with \( w' = \max_{i \in N}(p_i - t_i) + b, 0 \in q_N^{BC}(w', b) \) and \( 0 \leq q^* \).

**Proof of Lemma 4:** Suppose that \( w = c_m \). We know that \( q^* \in q_N^{BC}(c_m, v_w) \) since \( \pi_N^{BC}(q, c_m, v_w) = \pi^*(q) \). It follows from Lemma 3 that there exists a \( q \in q_N^{BC}(c_m, b) \) such that \( q \geq q^* \). The second argument is easy to check. \( \square \)

**Lemma 5** For any \( b, q_N^{BC}(w, b) \) is an upper-semicontinuous \( \text{(USC)} \) function in \( w \) and it is interval-valued.

**Proof of Lemma 5:** From Theorem 6, we know that \( \pi_N^{BC}(q, w, b) \) is a continuous function of \( q \) and \( w \). Then, it follows from the maximum theorem of Berge (1966) (the proof of this theorem can be

\[\text{Let } X \subseteq \mathbb{R}^k \text{ and } Y \subseteq \mathbb{R}^l \text{ be two non-empty sets. Let } F : X \to Y \text{ be a multi-valued function – i.e., a function from } X \text{ to } 2^Y \setminus \{\emptyset\}. F \text{ is called upper-semicontinuous in } x \in X \text{ if for every open neighborhood } V \text{ of } F(x) \text{ there exists an open neighborhood } U \text{ of } x \text{ with } F(\bar{x}) \subset V \text{ for every } \bar{x} \in U. F \text{ is called upper-semicontinuous if it is upper-semicontinuous in every } x \in X. \]
found in Hildenbrand (1974)) that $q^B(w, b)$ is USC with respect to $w$. Moreover, since $\pi_N^N(q, w, b)$ is a concave function of $q$, if there is a jump in $q^B(w, b)$ the entire interval should be in $q^B(w, b)$ as well. Hence, $q^B_N(w, b)$ is interval-valued.

Now, we will prove our first argument, which states that for all $v_w \leq b < \min_{i \in N} (p_i - t_i)$ there exists a $w$ such that $q^* \in q^B_N(w, b)$. From Lemma 4, we know that there are $w_1$ and $w_2$ such that $w_1 \leq w_2$, there exists a $q \in q^B_N(w_1, b)$ with $q \geq q^*$, and there exists a $q \in q^B_N(w_2, b)$ with $q \leq q^*$. Then, because of Lemma 5, it is a straightforward extension of the intermediate value theorem that there exists a $w_1 \leq w \leq w_2$ such that $q^* \in q^B_N(w, b)$. This proves our first argument.

It follows from Lemmas 2 and 3 that $w$ which satisfies $q^* \in q^B_N(w, b)$ is increasing with $b$. This proves our second argument and completes the proof.

**Theorem 10** For any coalition $S \subseteq N$, information sharing leads to better ordering and allocation decisions.

**Proof of Theorem 10:** Observe that in models with cooperation (under both the wholesale-price contracts and buy-back contracts) the expected profit function of a coalition consists of the the cost of placing the joint order, and the expected revenues from sales reduced by the allocation cost of the joint order (after demand forecast updates). Of those three parts, only the expressions for expected revenues use the demand forecasts. In what follows, we will show that for any coalition $S$, the true expected sales (hence true expected revenues) depend on $F_{yS_i}\omega$.

Consider a coalition $S$ and a joint order $q$ placed by that coalition. Let $y^S_S = (y_i)_{i \in S}$ be the demand signal vector observed by the members of coalition $S$ and let $\hat{\Omega} \subseteq \Omega$ be the set of states of the world which might trigger $y^S_S$. Observe that the demand signal of the retailers outside $S$ is irrelevant for coalition $S$, and that members of $S$ decide on an allocation of $q$ for each demand signal vector $y^S$. Let $A^S$ be the allocation decision of coalition $S$ after observing $y^S_S$. Then, the true expected sales of coalition $S$ for $y^S_S$ are given by

$$\sum_{\omega \in \Omega} P(\omega) \sum_{i \in S} A^S_i \int_0^{\bar{F}_{yS_i}(x)} dx = \sum_{i \in S} \int_0^{\bar{F}_{yS_i}(x)} P(\omega) \bar{F}_{w}(x) dx = \sum_{i \in S} \int_0^{\bar{F}_{yS_i}(x)} F_{yS_i}(x) dx$$

The last equality holds because there is one-to-one correspondence between $y^S_S$ and $F_{yS_i}$. Hence, we conclude that decisions based on expected sales calculated using $F_{yS_i}^i$ might result in worse performance for the coalition, since they might not reflect the reality.
Theorem 12 The game \((N, v^IS)\) has a non-empty core.

**Proof of Theorem 12:** Consider a model \(\Pi = (N, (X^i)_{i \in N}, (Y^i)_{i \in N}, t_w, (t_i)_{i \in N}, (p_i)_{i \in N}, (v_i)_{i \in N}, v_w, c_m)\). We will show that the associated game \((N, v^IS)\) has a non-empty core by using its relation with another game, \((N, w^{BC})\), associated with a model \(\Gamma\). We assume that the signals observed by the retailers in \(\Gamma\) are rich enough to update their knowledge to a level that they would achieve if they were sharing their information in model \(\Pi\), even if they do not form coalitions. We first show that the coalitions in model \(\Gamma\) perform better than coalitions in \(\Pi\). Then, we show that \((N, w^{BC})\) has a non-empty core, and that \(\text{Core}(w^{BC}) \subseteq \text{Core}(v^IS)\).

Consider the model defined by tuple \(\Gamma = (N, (X^i)_{i \in N}, (Y^i)_{i \in N}, t_w, (t_i)_{i \in N}, (p_i)_{i \in N}, (v_i)_{i \in N}, v_w, c_m)\) with \(\bar{Y}_i(\omega) = \cap_{i \in N} Y_j(\omega)\) for all \(\omega \in \Omega\) and \(i \in N\). In other words, every retailer receives the same signal, which gives him information that the retailers would obtain if they shared their signals in model \(\Pi\). Let \((N, w^{BC})\) denote the associated game in coalitional form. From Theorem 7, we know that \(\text{Core}(w^{BC})\) is non-empty.

Consider a coalition \(S\) and its order quantity \(q\). We remark that in model \(\Pi\) with information sharing, the joint information partition of each retailer \(i \in S\) is \(I_S\). Hence, for each \(\omega \in \Omega\), each retailer \(i \in S\) updates his demand information to \(y_S \in I_S\) with \(\omega \in y_S\), and expects demand distribution \(F^i_{y_S}\). Moreover, for each \(\omega \in y_S\), the retailers maximize the same revenue function,

\[
\bar{H}^\Pi_S(q, y_S, A) = - \sum_{i \in S} A_i (t_i - v_i) + b(q - \sum_{i \in S} A_i) + \sum_{i \in S} (p_i - v_i) \int_0^{A_i} F^i_{y_S}(x) dx.
\]

Therefore, we conclude that the retailers decide on an optimal allocation for each \(y_S \in I_S\). Let \(A^*(q, y_S)\) be the optimal allocation, and let \(G^\Pi_S(q, y_S) = \bar{H}^\Pi_S(q, y_S, A^*(q, y_S))\) be the corresponding maximal revenue of the coalition. Then, it is easy to check that the expected profit function of coalition \(S\) for order quantity \(q\) is given by

\[
\pi^\Pi_S(q) = -(w + t_w)q + E_{y_S \in I_S} [G^\Pi_S(q, \cdot)].
\]

We now repeat the same reasoning for model \(\Gamma\) without information sharing. Consider a coalition \(S\) and an order quantity \(q\). Since \(\bar{Y}_i(\omega) = \cap_{j \in N} Y_j(\omega)\) for all \(\omega \in \Omega\), there is a joint information partition for each retailer \(i \in S\), which is \(I_N\). Hence, for each \(\omega \in \Omega\), each retailer \(i \in S\) updates his demand information to \(y_N \in I_N\) with \(\omega \in y_N\) (whereas in model \(\Pi\) with information sharing it is \(y_S\)), and retailer \(i\) expects demand distribution \(F^i_{y_N}\). Moreover, for each \(\omega \in y_N\), the retailers
maximize the same revenue function,

$$H^\Gamma_S(q, y_N, A) = -\sum_{t \in S} A_t(t_i - v_i) + b(q - \sum_{t \in S} A_t) + \sum_{t \in S} (p_t - v_t) \int_0^{A_t} F_{y_N}(x) dx.$$ 

Therefore, we note that the retailers determine an optimal allocation for each $y_N \in I_N$, which we denote by $\bar{A}^*(q, y_N)$. Let $G^\Pi_S(q, y_N) = H^\Gamma_S(q, y_N, \bar{A}^*(q, y_N))$ be the corresponding maximal revenue of the coalition. The expected profit function of coalition $S$ for order quantity $q$ is given by

$$\pi^S(q) = -(w + t_w)q + E_{y_N \in I_N}[G^\Gamma_S(q, \cdot)].$$

We will first show that coalition $S$ in $\Pi$ cannot perform better than coalition $S$ in $\Gamma$. Note that $I_N$ is a refinement of $I_S$ – i.e., for all $y_N \in I_N$ there exists a $y_S \in I_S$ such that $y_N \subseteq y_S$. Consider $y_S \in I_S$ and an optimal allocation $A^*(q, y_S)$ for coalition $S$ given order $q$. Define, for all $y_N \in I_N$ with $y_N \subseteq y_S$, $\bar{A}(y_N) = A^*(q, y_S)$. Let us shortly denote $P(\omega \in y_N | \omega \in y_S)$ by $P(y_N | y_S)$. Then

$$G^\Pi_S(q, y_S) = H^\Gamma_S(q, y_N, A^*(q, y_S)) =$$

$$= -\sum_{t \in S} A_t^*(q, y_S)(t_i - v_i) + b(q - \sum_{t \in S} A_t^*(q, y_S)) + \sum_{t \in S} (p_t - v_t) \int_0^{A_t^*(q, y_S)} F_{y_N}(x) dx$$

$$= \sum_{y_N \in I_N : y_N \subseteq y_S} P(y_N | y_S) \left[-\sum_{t \in S} A_t(y_N)(t_i - v_i) + b \left(q - \sum_{t \in S} A_t(y_N)\right)\right]$$

$$+ \sum_{t \in S} (p_t - v_t) \int_0^{A_t(y_N)} F_{y_N}(x) dx$$

$$= \sum_{y_N \in I_N : y_N \subseteq y_S} P(y_N | y_S) H^\Gamma_S(q, y_N, \bar{A}(y_N))$$

$$\leq \sum_{y_N \in I_N : y_N \subseteq y_S} P(y_N | y_S) H^\Gamma_S(q, y_N, \bar{A}^*(q, y_N))$$

$$= \sum_{y_N \in I_N : y_N \subseteq y_S} P(y_N | y_S) G^\Gamma_S(q, y_N). \quad (A5)$$

The second equality holds because $I_N$ is a refinement of $I_S$ and, therefore, $\sum_{y_N \in I_N : y_N \subseteq y_S} P(y_N | y_S) = 1$, $\bar{A}(y_N) = A^*(q, y_S)$ for all $y_N \subseteq y_S$, and $\bar{F}_y = \sum_{y_N \in I_N : y_N \subseteq y_S} P(y_N | y_S) F_{y_N}^y$.

The third equality holds again because $I_N$ is a refinement of $I_S$ and $\bar{A}(y_N) = A^*(q, y_S)$ for all $y_N \subseteq y_S$, hence we can take $\sum_{y_N \in I_N : y_N \subseteq y_S} P(y_N | y_S)$ outside the integral. The inequality holds since $\bar{A}^*(q, y_N)$ is an optimal allocation.

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Let $q^*$ be the optimal order quantity of coalition $S$ in model $\Pi$. Then

$$v^{IS}(S) = \pi_S^{\Pi}(q^*) = -(w + t_w)q + E_{y_S \in I_S}[G_S^{\Pi}(q^*, \cdot)]$$

$$\leq -(w + t_w)q + E_{y_N \in I_N}[G_N^{\Gamma}(q^*, \cdot)] = \pi_S^{\Gamma}(q^*) \leq w^{BC}(S).$$

The first inequality follows from (A5) and because $I_N$ is a refinement of $I_S$. The last inequality holds because $v^{IS}(S)$ considers an optimal order quantity in $\Gamma$. It is easy to verify that $v^{IS}(N) = w^{BC}(N)$, because the retailers in the grand coalition have the same joint information partition, $I_N$, in both models. Therefore, we conclude that $Core(w^{BC}) \subseteq Core(v^{IS})$. As $Core(w^{BC}) \neq \emptyset$, this completes the proof. \hfill \Box
Appendix B - Using Duality to Find Core Allocations

Chen and Zhang (2006) show non-emptiness of the core for a two-stage stochastic inventory game. Following their approach, denote by $\bar{\Omega}_y$ the set of all possible outcomes of the random experiment governing the demand process of the retailers associated with demand signal $y$. Each retailer $i$ is facing a random demand $X^i_y$ with possible outcomes $X^i_y(\bar{\omega}_y)$ for each $\bar{\omega}_y \in \bar{\Omega}_y$. Because uncertainty is not resolved in the second stage of our model, we have to extend the model from Chen and Zhang by adding an additional stage, where random demand faced by the retailers depends on the demand signal. Thus, we rewrite $H^C_S(q, y, A)$ as

$$H^C_S(q, y, A) = -\sum_{i \in S} t_i A_i + E_{\bar{\Omega}_y}[R(A, X^S_y(\cdot))],$$

where $X^S_y = (X^i_y)_{i \in S}$ and $R(A, X^S_y(\omega_y))$ denotes the revenue for given inventory allocation and demand realization:

$$R(A, X^S_y) = \max \sum_{i \in S} p_i s_i + \sum_{i \in S} v_i o_i \quad (B1a)$$

s.t. $s_i + o_i \leq A_i, \quad i \in S \quad (B1b)$

$$o_i + x_i \geq A_i, \quad i \in S \quad (B1c)$$

$$s_i \leq x_i, \quad i \in S \quad (B1d)$$

$$o_i, s_i \geq 0, \quad i \in S. \quad (B1e)$$

Here, $s_i$ denotes actual sales at retailer $i$, $o_i$ denotes excess inventory at retailer $i$, and $x_i = X^i_y(\omega_y)$. The first constraint states that total sales and remaining inventory cannot exceed initial allocation, while the second constraint states that the remaining inventory cannot be lower than the amount obtained by subtracting the realized demand from the retailer’s inventory allocation. The last constraint states that the sales cannot exceed the demand. Thus, in the last stage of our model, we maximize function $R(A, X^S_y)$; in the second stage we maximize the profit after the signal is observed and the warehouse costs are paid, $H^C_S(q, y, A)$; and in the first stage, we maximize the expected profit of the coalition. Let us denote by $\alpha_i(\omega^y), \beta_i(\omega^y)$, and $\gamma_i(\omega^y)$ dual variables associated with constraints (B1b) - (B1d) for outcome $\omega^y$, and let $\delta(y)$ be dual variable associated with the constraint in problem (4) when demand signal is $y$. Then, we can write the dual problem of our three-stage
problem as

$$
\begin{align*}
\min & \quad E_Y \left[ E_{\bar{\Omega} y} \left( \sum_{i \in S} X_i \cdot (\beta_i + \gamma_i) \right) \right] \\
\text{s.t.} & \quad E_Y \left[ \delta(\cdot) \right] \leq w + t_w \\
& \quad E_{\bar{\Omega} y} [\alpha_i(\cdot) - \beta_i(\cdot)] \leq t_i + \delta(y), \quad i \in S, y \in Y \\
& \quad \alpha_i(\bar{\omega}^y) + \gamma_i(\bar{\omega}^y) \geq p_i, \quad i \in S, \bar{\omega}^y \in \bar{\Omega}^y, y \in Y \\
& \quad \alpha_i(\bar{\omega}^y) - \beta_i(\bar{\omega}^y) \geq v_i, \quad i \in S, \bar{\omega}^y \in \bar{\Omega}^y, y \in Y \\
& \quad \alpha_i(\bar{\omega}^y), \beta_i(\bar{\omega}^y), \gamma_i(\bar{\omega}^y) \geq 0, \quad i \in S, \bar{\omega}^y \in \bar{\Omega}^y, y \in Y.
\end{align*}
$$

To further simplify the expressions, denote

$$
\eta_i(\bar{\omega}^y) = \alpha_i(\bar{\omega}^y) - \beta_i(\bar{\omega}^y), \quad \text{and} \quad \vartheta_i(\bar{\omega}^y) = \beta_i(\bar{\omega}^y) + \gamma_i(\bar{\omega}^y)
$$

for all \( i \in N \). Then, the dual becomes

$$
\begin{align*}
\min & \quad E_Y \left[ E_{\bar{\Omega} y} \left( \sum_{i \in S} X_i \cdot \vartheta_i(\cdot) \right) \right] \tag{B2a} \\
\text{s.t.} & \quad E_Y [\delta(\cdot)] \leq w + t_w \tag{B2b} \\
& \quad E_{\bar{\Omega} y} [\eta_i(\cdot)] \leq t_i + \delta(y), \quad i \in S, y \in Y \tag{B2c} \\
& \quad \eta_i(\bar{\omega}^y) + \vartheta_i(\bar{\omega}^y) \geq p_i, \quad i \in S, \bar{\omega}^y \in \bar{\Omega}^y, y \in Y \tag{B2d} \\
& \quad \eta_i(\bar{\omega}^y) \geq v_i, \quad i \in S, \bar{\omega}^y \in \bar{\Omega}^y, y \in Y \tag{B2e} \\
& \quad \vartheta_i(\bar{\omega}^y) \geq 0, \quad i \in S, \bar{\omega}^y \in \bar{\Omega}^y, y \in Y. \tag{B2f}
\end{align*}
$$

The interpretation of the dual is similar to that in Chen and Zhang (2006). The primal problem corresponds to the model in which the retailers purchase items directly from the manufacturer, and then allocate the orders at the warehouses (after observing demand signal) in a profit-maximizing way. For the dual model, consider the following scenario. The retailers can purchase the units directly from the warehouse, which charges them a unit price \( \delta(y) \), based on the demand signal. The retailers would benefit by ordering directly from the manufacturer, unless the cost charged by the warehouse is lower than the manufacturer’s cost augmented by the transportation cost to the warehouse, which gives the first constraint. However, the retailers outsource the ordering and allocations to a third party, which charges them depending on the realization of the uncertainty. The third party charges retailer \( i \) a unit price \( \eta_i(\bar{\omega}^y) \). To prevent the retailers to order directly from the warehouse, the amount charged by this third party should not exceed the amount charged by the warehouse augmented by the transportation cost from the warehouse to the retailer; this gives the second constraint. The third party wants to charge the retailers as much as possible. Denote by
\( \vartheta_i(\bar{\omega}^y) \) the margin left to the retailer \( i \). The third party wants to minimize the total profit left at the retailers, which gives our objective function. The margin is obtained as the difference between the retail price and the amount charged by the third party, which gives the third constraint. To prevent inefficient outcomes, the third party should charge the retailers more than what they can get by salvaging their product, which gives the last constraint.

Suppose that \((\delta^*(y), \eta^*(\bar{\omega}^y), \vartheta^*(\bar{\omega}^y))\) is the optimal solution of the dual with \( S = N \). Then, 

\[
\varphi_i = E_Y \left[ E_{\bar{\Omega}^y} \left[ X^i_y(\cdot)\vartheta^*_i(\cdot) \right] \right]
\]

is a core allocation, as follows: observe that, by definition,

\[
\sum_{i \in N} \varphi_i = E_Y \left[ \sum_{i \in N} E_{\bar{\Omega}^y} \left[ X^i_y(\cdot)\vartheta^*_i(\cdot) \right] \right],
\]

which is the optimal value of the dual. Following the arguments from Chen and Zhang (2006), we can conclude that the optimal value of the dual coincides with the optimal value of the primal problem, hence

\[
E_Y \left[ \sum_{i \in N} E_{\bar{\Omega}^y} \left[ X^i_y(\cdot)\vartheta^*_i(\cdot) \right] \right] = \max_q \pi^C_{N}(q).
\]

Thus, \( \varphi \) allocates the entire profit. Next, because \((\delta^*(y), \eta^*(\bar{\omega}^y), \vartheta^*(\bar{\omega}^y))\) is a feasible solution to the dual with \( S = N \), its restriction to the set \( S \) gives us a feasible solution for the subgame. The corresponding value of the dual is

\[
E_Y \left[ \sum_{i \in S} E_{\bar{\Omega}^y} \left[ X^i_y(\cdot)\vartheta^*_i(\cdot) \right] \right] = \sum_{i \in S} \varphi_i.
\]

Because the optimal value of the primal for \( S \) corresponds to the optimal value of the dual, we have

\[
\max_q \pi^C_{S}(q) \leq \sum_{i \in S} \varphi_i,
\]

which shows that \( \varphi \) belongs to the core. Thus, we can construct a core allocation for our problem.

If we consider the model with buy-backs and cooperating retailers, the only change that needs to be made in the dual problem is replacing constraints (B2b)–(B2c) with

\[
E_Y[\delta(\cdot)] \leq w + t_w + b
\]

\[
E_{\bar{\Omega}^y}[\eta_i(\cdot)] \leq t_i + \delta(y) - b, \quad i \in S, y \in Y
\]

This change can be interpreted as follows: because the warehouse is now offering to purchase all unsold items at price \( b \), it can increase its price, \( \delta(y) \), by at most the same amount. This gives the first constraint. As the warehouse price can go up by \( b \), and it is a part of the upper bound for the third-party price, \( \eta_i(\bar{\omega}^y) \), the upper bound for \( \eta_i(\bar{\omega}^y) \) is reduced by \( b \). This gives the second constraint.