Appendix A

Theorem 3. In an inventory-sharing game with \( n \) symmetric retailers facing strictly increasing and independent distribution functions, there is an \( M > 0 \) such that \( \delta^*_n \) is decreasing in \( n \) for \( n \geq \hat{n} \), where \( \hat{n} = \min\{ n \in \mathbb{Z} : nX^d \geq M \} \).

Proof of Theorem 3: In order to prove this theorem, we first introduce the following notation: let \( F^m(y) = P\{\sum_{i=1}^{m} D_i \leq y\} \), \( \hat{F}^m(y) = P\{\frac{1}{m} \sum_{i=1}^{m} D_i \leq y\} \), and \( E[D_i] = \mu \). Note that \( F^m(y) = \hat{F}^m(\frac{y}{m}) \). We will also need the following lemmas.

Lemma A1. In an inventory-sharing game with symmetric retailers facing strictly increasing and independent distribution functions, a retailer defecting from strategy \((X^d, \bar{H}_i, \bar{E}_i)\) maximizes her benefit from defection if she orders \( X^d \).

Proof of Lemma A1: If we have \( n \) symmetric retailers, the dual price of retailer \( i \)'s residual will be either 0 or \( p \), depending on the amount she is sharing with the others. For example, if \( \sum_{j \neq i}(\bar{E}_j - \bar{H}_j) = k > 0 \), the retailers other than \( i \) need \( k \) additional units of products. Then, retailer \( i \) will receive \( p \) per unit if \( 0 < \bar{H}_i < k \), while she will get nothing otherwise. More formally, retailer \( i \)'s total expected profit when she orders \( X^d \) and other retailers order \( X^d_{-i} \) is given by

\[
J^d_i(X^d_i | X^d_{-i}) = rE[\min\{X^d_i, D_i\}] + vE[H_i] - cX^d_i + \frac{\mu}{n} \int_0^{\infty} \left( \int_X (X^d_i - u) f(u) \, du \right) \, dk + \frac{p}{n} \int_0^{\infty} \left( \int_X (X^d_i - k) f(u) \, du \right) \, dk,
\]

where \( f^{n-1}((n-1)X^d_i + y) \) is the probability density when the residual demand (resp., inventory)
for the remaining \((n - 1)\) retailers is \(y > 0 \) (resp., \((-y) > 0\)), and its first derivative is given by

\[
(J_i^{df})'(X_i|x_i|X_{-i}) = r - c - (r - v)F(X_i) + p \int_0^\infty [F(X_i) - F(X_i - k)] f^{n-1} \left((n - 1)X^d + k\right) dk - \\
p \int_0^\infty [F(X_i + k) - F(X_i)] f^{n-1} \left((n - 1)X^d - k\right) dk - \\
p \int_0^\infty k f'(X_i - k) f^{n-1} \left((n - 1)X^d + k\right) - f(X_i + k) f^{n-1} \left((n - 1)X^d - k\right) dk,
\]

(A1)

Retailer \(i\) can increase her profit if she deviates whenever her dual price is zero. In other words, she maximizes her profit if she withholds part of her residual inventory/demand to make it lower than the total residual demand/inventory from other retailers. Under this kind of strategy, her total expected profit will be increased to

\[
J_i^{df}(X_i|x_i|X_{-i}) = rE[\min\{X_i, D_i\}] + vE[H_i] - cX_i + p \int_0^\infty (n - 1)X^d + k \int_{X_i - k}^{X_i} (X_i - u) f(u) du dk + \\
p \int_0^\infty (n - 1)X^d - k \int_{X_i}^{X_i + k} (u - X_i) f(u) du dk + \\
p \int_0^\infty (n - 1)X^d + k \int_{X_i}^{X_i + k} F(X_i - k) dk + p \int_0^\infty (n - 1)X^d - k [1 - F(X_i + k)] dk,
\]

and its derivatives are

\[
(J_i^{df})'(X_i|x_i|X_{-i}) = r - c - (r - v)F(X_i) + p \int_0^\infty [F(X_i) - F(X_i - k)] f^{n-1} \left((n - 1)X^d + k\right) dk - \\
p \int_0^\infty [F(X_i + k) - F(X_i)] f^{n-1} \left((n - 1)X^d - k\right) dk,
\]

(A2)

\[
(J_i^{df})''(X_i|x_i|X_{-i}) = -tf(X_i) - p \int_0^\infty \left[f(X_i - k) f^{n-1} \left((n - 1)X^d + k\right) - f(X_i + k) f^{n-1} \left((n - 1)X^d - k\right)\right] dk < 0.
\]

Because all demands follow an identical distribution, it follows from (A1) and (A2) that

\[
[(J_i^{df})' - (J_i^{df})'](X_i|x_i|X_{-i}) = p \int_0^\infty \left[f(X_i - k) f^{n-1} \left((n - 1)X^d + k\right) - f(X_i + k) f^{n-1} \left((n - 1)X^d - k\right)\right] dk
\]

\[
= E \left[ X_i - D_i \right] \sum_{m=1}^{n} D_m = (n - 1)X^d + X_i = \frac{n - 1}{n} \left( X_i - X^d\right).
\]

Recall that \(X^d = \arg \max J_i^{df}(X_i|x_i|X_{-i})\), and consequently \((J_i^{df})'(X^d|x_i|X_{-i}) = 0\). This implies

\[
(J_i^{df})'(X^d|x_i|X_{-i}) = (J_i^{df})'(X^d|x_i|X_{-i}) + [(J_i^{df})'(X^d|x_i|X_{-i}) - (J_i^{df})'(X^d|x_i|X_{-i})] = 0 + \frac{n - 1}{n} (X^d - X^d) = 0.
\]

Since \(J_i^{df}(X_i|x_i|X_{-i})\) is a concave function, the optimal ordering decision when player \(i\) defects, \(X_i^{df}\), should satisfy \((J_i^{df})'(X_i^{df}|X_{-i}) = 0\). Thus, \(X_i^{df} = X^d\), and a retailer contemplating a defection maximizes her profit if she orders at the decentralized optimal level. \(\square\)
Lemma A2. In an inventory-sharing game with \( n \) symmetric retailers and strictly increasing demand distribution function, the expected profit for each retailer, \( J^d(X^d(n), n) \), is increasing in \( n \), where \( X^d(n) \) is the NE ordering decision for each retailer in the decentralized system.

Proof of Lemma A2: Consider a game with \( n + 1 \) symmetric retailers, and let \( S \) be any \( n \)-members subset of these retailers. In terms of cooperative game theory, the value of the coalition \( S \) corresponds to the profit generated by its members; because the retailers are symmetric, it can be written as \( V_S^* = nJ^d(X, n) \), where \( J^d(X, n) \) denotes the expected profit generated by an arbitrary retailer in a game with \( n \) symmetric retailers under dual allocations. However, in an \((n+1)\)-retailer game with dual allocations, each retailer will receive a payoff \( J^d(X, n+1) \). Because dual allocations belong to the core, we must have \( nJ^d(X, n+1) > V_S^* = nJ^d(X, n) \). It is then straightforward that \( J^d(X^d(n+1), n+1) \geq J^d(X^d(n), n+1) \geq J^d(X^d(n), n) \).

We can now prove the theorem. Consider the model with \( n \) symmetric retailers and suppose that there were no prior defections. That is, each retailer orders \( X^d \) and shares her entire residuals. Recall that we have shown in Lemma A1 that defecting retailers maximize their profit if they order \( X^d \) and deviate in the amount they share with others. Under demand realization \( D \), let \( \bar{P}_{i, i}^{def}(X^d, D, n) \) denote the highest payoff that retailer \( i \) can generate if she defects in a game with \( n \) players, while the other retailers cooperate, and recall that \( P_i^d(X^d, D, n) \) is her profit in the current period if she shares all of her residuals. After defection, she will receive \( J_i(X_1) \) in all subsequent periods. Thus, a possible deviation by player \( i \) is deterred if her discount factor satisfies

\[
\bar{P}_{i, i}^{def}(X^d, D, n) + \frac{\delta}{1-\delta} J_i(X_1) < \frac{\delta}{1-\delta} J_i^d(X^d, n) + P_i^d(X^d, D, n), \forall D, \tag{A3}
\]

where \( J_i^d(X^d, n) \) denotes the payoff that retailer \( i \) receives when \( n \) retailers use dual allocations, order \( X^d \), and share their entire residuals. It is easy to verify that (A3) holds whenever

\[
\delta > \delta_{i,n} = \frac{1}{1 + \frac{J_i^d(X^d, n) - J_i(X_1)}{\sup_D \{ P_{i, i}^{def}(X^d, D, n) - P_i^d(X^d, D, n) \}}} \tag{A4}
\]

Note that the upper bound of the extra profit that one can get out of deviation, \( \sup_D \{ \bar{P}_{i, i}^{def}(X^d, D, n) - P_i^d(X^d, D, n) \} \), can be obtained by comparing two cases: (i) the extra profit generated when \( D_i = 0 \) and the total residual demand of the remaining retailers is slightly below \( X^d \); and (ii) the extra profit generated when \( D_{\sim i} = 0 \) and \( D_i \) is slightly above \( nX^d \). In the first case, this profit is \( pX^d \); in the second case, this profit would be \( p(n-1)X^d \), assuming that demand can achieve values above \( nX^d \). However, note that in most real-life situations there is an \( M > 0 \) such that \( P(D_i > M) \) is negligible (if demand distribution has a finite support with upper bound \( \bar{D} \), then \( M = \bar{D} \)), and the maximum benefit from defection is \( p(M - X^d) \). Let us denote \( \hat{n} = \min \{ n : nX^d \geq M \} \). Then, whenever \( n \geq \hat{n} \), it implies that \( \sup_D \{ \bar{P}_{i, i}^{def}(X^d, D, n) - P_i^d(X^d, D, n) \} = \max \{ pX^d, p(M - X^d) \} \), and (A4) corresponds to

\[
\delta > \delta_{i,n} = \frac{p \max \{ X^d, M - X^d \}}{p \max \{ X^d, M - X^d \} + J_i^d(X^d, n) - J_i(X_1)}.
\]
Because the players are symmetric, let $\delta_n = \delta_{i,n}$. Since $J_i(X_1)$ does not depend on $n$ and we showed in Lemma A2 that $J_i^d(X^d, n)$ increases with $n$, $\delta_n$ is decreasing in $n$. Finally, let $\delta_n^* = \delta_n$. 

**Proposition 2.** In an inventory-sharing game with $n$ symmetric retailers and strictly increasing distribution function $F(\cdot)$, the asymptotic behavior of the equilibrium ordering quantity can be described by

$$\lim_{n \to \infty} X^d(n) = \begin{cases} 
\mu, & \text{if } t = 0 \text{ or } \frac{r-c-p}{t} \leq F(\mu) \leq \frac{r-c}{t}, \\
\sup\{x : F(x) < \frac{r-c}{t}\} & \text{if } F(\mu) > \frac{r-c}{t}, \\
\inf\{x : F(x) > \frac{r-c-p}{t}\} & \text{if } F(\mu) < \frac{r-c-p}{t}.
\end{cases}$$

**Proof of Proposition 2:** When each retailer orders $X^d$, the total expected profit for each of them can be determined by

$$J(X^d) = rE[\min\{X^d, D\}] + vE[H] - cX^d + p\int_0^\infty k f(X^d - k) \left[1 - \hat{F}^{n-1} \left(X^d + \frac{k}{n-1}\right)\right] dk + p\int_0^\infty k f(X^d + k) \hat{F}^{n-1} \left(X^d - \frac{k}{n-1}\right) dk$$

$$= (r-c)X^d - (r-v)X^dF(X^d) - \int_0^\infty yf(y)dy + p\int_0^\infty k f(X^d - k) \left[1 - \hat{F}^{n-1} \left(X^d + \frac{k}{n-1}\right)\right] dk + p\int_0^\infty k f(X^d + k) \hat{F}^{n-1} \left(X^d - \frac{k}{n-1}\right) dk.$$ 

If we let $\sigma^2 = Var[D_i]$, then by the Central Limit Theorem (CLT) we have

$$\lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^m D_i \sim N \left(\mu, \frac{\sigma^2}{m}\right).$$

Suppose first that $X^d > \mu$. Then, we have $\lim_{n \to \infty} [1 - \hat{F}^{n-1}(X^d + \frac{k}{n-1})] = 0$ and $\lim_{n \to \infty} \hat{F}^{n-1}(X^d - \frac{k}{n-1}) = 1$, hence the derivative of $J(\cdot|X^d_{-i})$ evaluated at $X^d$ becomes

$$J'(X^d|X^d_{-i}) = r - c - (r-v)F(X^d) - p + pF(X^d) = -(c-v) + t[1 - F(X^d)],$$

which is a decreasing function of $X^d$. Thus, if $t = 0$ or $F(\mu) \geq 1 - \frac{c-v}{t} = \frac{r-c-p}{t}$, then $J'(X^d|X^d_{-i}) \leq 0$ for any $X^d \in (\mu, \infty)$, and the retailer maximizes her profit by choosing $X^d \to \mu^+$. Otherwise, $X^d = \inf\{x : F(x) > \frac{r-c-p}{t}\}$ is an optimal solution within $(\mu, \infty)$.

If $X^d < \mu$, $\lim_{n \to \infty} [1 - \hat{F}^{n-1}(X^d + \frac{k}{n-1})] = 1$ and $\lim_{n \to \infty} \hat{F}^{n-1}(X^d - \frac{k}{n-1}) = 0$. The derivative of $J(\cdot|X^d_{-i})$ evaluated at $X^d$ becomes

$$J'(X^d|X^d_{-i}) = r - c - (r-v)F(X^d) + pF(X^d) = (r-c) - tF(X^d),$$

which is again a decreasing function of $X^d$. In this case, if $F(\mu) \leq \frac{r-c}{t} \text{ or } t = 0$, then $J'(X^d|X^d_{-i}) \geq 0$ for any $X^d \in (\infty, \mu)$, and the retailer maximizes her profit by choosing $X^d \to \mu^-$. Otherwise, $X^d = \sup\{x : F(x) < \frac{r-c}{t}\}$ is an optimal solution within $(-\infty, \mu)$. 

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From the above, we can conclude that whenever $F(\mu) \in \left[\frac{r-c-p}{t}, \frac{r-c}{t}\right]$ or $t = 0$, the retailer should select $X^d \to \mu$. Otherwise, because $\frac{r-c-p}{t} \leq \frac{r-c}{t}$, any local optimum is also a global optimum whenever $F(\mu) \notin \left[\frac{r-c-p}{t}, \frac{r-c}{t}\right]$.

**Corollary 1.** In an inventory-sharing game with $n$ symmetric retailers and strictly increasing distribution function $F(\cdot)$, the following relationships hold when $n$ is large:

1. When $t > 0$: if $F(\mu) > \frac{r-c}{t}$, then $X^1 \leq X^d(n) < \mu$; if $F(\mu) < \frac{r-c-p}{t}$, then $\mu < X^d(n) \leq X^1$.

2. When $t = 0$: if $F(\mu) > \frac{r-c}{t}$, then $X^1 \leq X^d(n) = \mu$; if $F(\mu) < \frac{r-c}{t}$, then $X^1 \geq X^d(n) = \mu$.

**Proof of Corollary 1:** Suppose first that $t > 0$. If $F(\mu) > \frac{r-c}{t}$, it follows from Proposition 2 that $\lim_{n \to \infty} X^d(n) = \sup\{x : F(x) < \frac{r-c}{t}\}$. This implies that $F(X^d) \leq \frac{r-c}{t} < F(\mu)$, hence $X^d < \mu$. On the other hand, when there is no cooperation among the retailers, the optimal ordering level $X^1$ can be determined by the newsvendor model, $F(X^1) = \frac{r-c}{t}$. Recall that we assume $p=r-v-t \geq 0$, which implies $r-v \geq t$, therefore $F(X^1) \leq F(X^d)$, and $X^1 \leq X^d$.

If, on the other hand, $F(\mu) < \frac{r-c-p}{t}$, then $\lim_{n \to \infty} X^d(n) = \inf\{x : F(x) > \frac{r-c-p}{t}\}$. This implies that $F(\mu) < \frac{r-c-p}{t} \leq F(X^d)$, hence $\mu < X^d$. Consequently, $F(X^1) = \frac{r-c}{t} \geq \frac{r-c-p}{t} = F(X^d)$, so $X^1 \geq X^d$.

When $t = 0$, each retailer orders the expected demand value, and the result is straightforward.

**Theorem 4.** In an inventory-sharing game with $n$ symmetric retailers and strictly increasing distribution function $F(\cdot)$, $\delta^*_n \to \delta^*_\infty > 0$. More specifically, let $M$ be as defined in Theorem 3, and let $\xi(x) = \int_0^x yf(y)dy$ and $\varrho(x) = p \max\{x, M-x\}$. Then,

$$
\delta^*_\infty = \begin{cases} 
\frac{\varrho(\mu)}{\varrho(\mu) + (r-c-tF(\mu))\mu - (r-v)\xi(X^1)}, & \text{if } \frac{r-c-p}{t} \leq F(\mu) \leq \frac{r-c}{t} \text{ or } t = 0; \\
\frac{\varrho(X^d)}{\varrho(X^d) + t\xi(X^d) - (r-v)\xi(X^d)}, & \text{if } F(\mu) > \frac{r-c}{t} \text{ and } X^d = \sup\{x : F(x) < \frac{r-c}{t}\}; \\
\frac{\varrho(X^d)}{\varrho(X^d) + t(\xi(X^d) - \mu) - (r-v)(\xi(X^1) - \mu)}, & \text{if } F(\mu) < \frac{r-c-p}{t} \text{ and } X^d = \sup\{x : F(x) > \frac{r-c-p}{t}\}.
\end{cases}
$$

**Proof of Theorem 4:** Recall that the lower bound of $\delta_n$ satisfies

$$
\delta_n^* = \frac{p \max\{X^d, M - X^d\}}{p \max\{X^d, M - X^d\} + J^d_i(X^d, n) - J_i(X_1)} = \frac{\rho(X^d)}{\rho(X^d) + J^d_i(X^d, n) - J_i(X_1)} \forall i. \quad (A5)
$$

In addition, in the model without cooperation, each retailer’s profit is maximized at $X^1 = F^{-1}\left(\frac{r-c}{r-v}\right)$, and equals

$$
J^1(X^1) = (r-v) \int_0^{X^1} yf(y)dy = (r-v)\xi(X^1). \quad (A6)
$$
If $X^d = \mu$, it follows from the CLT that $\lim_{n \to \infty} 1 - \hat{F}^{n-1}(X^d + \frac{k}{n-1}) = \lim_{n \to \infty} \hat{F}^{n-1}(X^d - \frac{k}{n-1}) = \frac{1}{2}$, which implies

$$J^d_i(X^d, n) = (r - c)X^d - (r - v) \left[ X^d F(X^d) - \int_0^{X^d} y f(y) dy \right]$$

$$+ p \int_0^\infty k f(X^d - k) \left[ 1 - \hat{F}^{n-1} \left( X^d + \frac{k}{n-1} \right) \right] dk + p \int_0^c k f(X^d + k) \hat{F}^{n-1} \left( X^d - \frac{k}{n-1} \right) dk$$

$$= (r - c)\mu - (r - v) \left[ \mu F(\mu) - \int_0^{\mu} y f(y) dy \right] + \frac{p}{2} \int_0^\infty k f(\mu - k) dk + p \int_0^c k f(\mu + k) dk$$

$$= [r - c - t F(\mu)] \mu + t \int_0^{\mu} y f(y) dy$$

$$= [r - c - t F(\mu)] \mu + t \xi(\mu) \quad (A7)$$

By substituting (A6) and (A7) into (A5), we obtain

$$\delta^*_\infty = \frac{\rho(\mu)}{\rho(\mu) + [r - c - t F(\mu)] \mu + t \xi(\mu) - (r - v)\xi(X^1)}.$$ 

If $X^d = \sup \{ x : F(x) < \frac{r - c}{t} \} < \mu$, we have $\lim_{n \to \infty} 1 - \hat{F}^{n-1}(X^d + \frac{k}{n-1}) = 1$ and $\lim_{n \to \infty} \hat{F}^{n-1}(X^d - \frac{k}{n-1}) = 0$, hence

$$J^d_i(X^d, n) = (r - c)X^d - (r - v) \left[ X^d F(X^d) - \int_0^{X^d} y f(y) dy \right]$$

$$+ p \int_0^\infty k f(X^d - k) \left[ 1 - \hat{F}^{n-1} \left( X^d + \frac{k}{n-1} \right) \right] dk + p \int_0^\infty k f(X^d + k) \hat{F}^{n-1} \left( X^d - \frac{k}{n-1} \right) dk$$

$$= (r - c)X^d - (r - v) \left[ X^d F(X^d) - \int_0^{X^d} y f(y) dy \right] + \frac{p}{2} \int_0^\infty k f(X^d) dk$$

$$= t \int_0^{X^d} y f(y) dy$$

$$= t \xi(X^d) \quad (A8)$$

By substituting (A6) and (A8) into (A5), we obtain

$$\delta^*_\infty = \frac{\rho(X^d)}{\rho(X^d) + t \xi(X^d) - (r - v)\xi(X^1)}.$$ 

Finally, if $X^d = \inf \{ x : F(x) > \frac{r - c - 1}{t} \} > \mu$, we have $\lim_{n \to \infty} 1 - \hat{F}^{n-1}(X^d + \frac{k}{n-1}) = 0$ and
Proof of Proposition 5: 
It cannot be achieved.

\[ \lim_{n \to \infty} \hat{F}^{n-1}(X^d - \frac{k}{n-1}) = 1, \text{ hence} \]

\[ J_i^d(X^d, n) = (r - c)X^d - (r - v) \left[ X^d F(X^d) - \int_{0}^{X^d} y f(y) dy \right] \]

\[ + p \int_{0}^{\infty} k f(X^d - k) \left[ 1 - \hat{F}^{n-1} \left( X^d + \frac{k}{N-1} \right) \right] dk + p \int_{0}^{\infty} k f(X^d + k) \hat{F}^{n-1} \left( X^d - \frac{k}{n-1} \right) dk \]

\[ = (r - c)X^d - (r - v) \left[ X^d F(X^d) - \int_{0}^{X^d} y f(y) dy \right] + p \int_{0}^{\infty} k f(X^d + k) dk \]

\[ = p\mu + t \int_{0}^{X^d} y f(y) dy \]

\[ = p\mu + t\xi(X^d) \quad \text{(A9)} \]

By substituting (A6) and (A9) into (A5), we obtain

\[ \delta^*_\infty = \frac{\rho(X^d)}{\rho(X^d) + p\mu + t\xi(X^d) - (r - v)\xi(X^d)}. \]

\[ \square \]

**Proposition 5.** If \( n \) retailers face i.i.d. demand distributions and differ only in their material costs (that is, \( r_i = r_j = r, v_i = v_j = v, t_{ij} = t_{ji} = t \) for \( i, j \in \{1, \ldots, n\} \)), a first-best outcome cannot be achieved.

**Proof of Proposition 5:** Retailers have the same demand distribution \( F(\cdot) \), price \( r \), salvage value \( v \), transshipping cost \( t \), and unit profit from transshipment \( p = r - v - t \). Denote \( X = \sum_j X_j, X_{-i} = \sum_{j \neq i} X_j \) and let \( f^m \) the p.d.f of \( mD_i \). It can be verified that

\[ \frac{\partial J_i^d}{\partial X_i} - \frac{\partial J_i^n}{\partial X_i} = p \int_{0}^{\infty} k f(X_i - k) f^{n-1}(X_{-i} + k) dk - p \int_{0}^{\infty} k f(X_i + k) f^{n-1}(X_{-i} - k) dk \]

\[ = p E[X_i - D_i | X = D] f^n(X) \]

Denote \( O_i = \left( \frac{\partial J_i^d}{\partial X_i} - \frac{\partial J_i^n}{\partial X_i} \right) \big|_{X^o} \). Achieving first best requires \( O_i = 0 \) for all \( i \). However, for any \( i \neq j \),

\[ O_i - O_j = p f^n(X) E[X_i^n - X_j^n + D_j - D_i | D = X] \]

\[ = p f^n(X) \left[ X_i^n - X_j^n + E[D_j - D_i | D = X] \right] \]

\[ = p f^n(X) (X_i^n - X_j^n) \]

It therefore requires \( X_i^n = X_j^n \), \( \forall i, j \). This is obviously not true given that each \( X_i^n \) has to satisfy its FOC with a different \( c_i \):

\[ \frac{\partial J_i^n}{\partial X_i^n} = r - c_i + (r - v) F(X_i^n) + p Pr\{D_i \leq X_i^n, D > X^n\} - p Pr\{D_i \geq X_i^n, D < X^n\} = 0 \]

\[ \square \]

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Theorem 5. Suppose that all retailers participate in inventory sharing and \( J^n(X) \) is unimodal. Then, the eviction contract \((X_t(\hat{h}_{t-1}), H_t(\hat{h}_{t-1}), E_t(\hat{h}_{t-1}), d_t(\hat{h}_t), B)\) is a contract that induces a first-best solution if the retailers’ ordering strategies, \( X_t \), are given by

\[ X_t(\hat{h}_{t-1}|Z_t = Z^k) = X^k, \]

all coalition members share their entire residuals, the evicted members share nothing, the discretionary transfer payments are

\[
d_{it}(\hat{h}_t) = \begin{cases} 
\frac{\Delta_{it}(\hat{h}_t)}{\sum_{i \in I_t^+} \Delta_{it}(\hat{h}_t)} \times \sum_{i \in I_t^-} (-\Delta_{jt}(\hat{h}_t)) & i \in I_t^+ \\
\Delta_{it}(\hat{h}_t) & i \in I_t^-,
\end{cases}
\]

where

\[
\Delta_{it}(\hat{h}_t) = \frac{1}{1 - \delta_i} \left[ J^n_i(X^n) - \delta J^1_i(X^1) \right] - J^n_i(X_t, H_t, E_t),
\]

\[ I_t^+ = \{ i : \Delta_{it}(\hat{h}_t) > 0 \} \text{ and } I_t^- = \{ i : \Delta_{it}(\hat{h}_t) \leq 0 \}, \]

and the one-time contract activation bonus is given as

\[
B_i = \begin{cases} 
\frac{\Lambda_i}{\sum_{i \in K^-} \Lambda_i} \times \sum_{i \in K^+} (-\Lambda_i) & i \in K^- \\
\Lambda_i & i \in K^+,
\end{cases}
\]

where

\[
\Lambda_i = \frac{1}{1 - \delta_i} (J^n_i(X^n) - J^n_i(X^1)), \quad K^+ = \{ i : \Lambda_i > 0 \} \text{ and } K^- = \{ i : \Lambda_i \leq 0 \}.
\]

Proof of Theorem 5: The eviction contract described in Theorem 5 will be an optimal contract if it satisfies the following constraints:

1. Participation constraint – each retailer is better off if she adopts the contract;

2. Early adoption constraint – each retailer prefers to adopt the contract in the current period than in the later period;

3. Continuation constraints – each retailer is better off if she does not deviate in any period.

We now show that the eviction contract satisfies all three constraints.

Participation constraint: If retailer \( i \) adopts the contract in period 1, her infinite horizon discounted payoff is given by

\[
B_i + \sum_{t=1}^{\infty} \delta_i^{t-1} J^n_i(X^n) = B_i + \frac{1}{1 - \delta_i} J^n_i(X^n).
\]
If the contract is not adopted and each retailer orders the individually optimal quantity (under the dual allocation rule), her payoff is

$$\sum_{t=1}^{\infty} \delta_t^{t-1} J_i^n(X^d_t) = \frac{1}{1 - \delta_i} J_i^n(X^d_t).$$

The participation constraint is satisfied if

$$B_i + \frac{1}{1 - \delta_i} J_i^n(X^n_t) \geq \frac{1}{1 - \delta_i} J_i^n(X^d_t).$$

First, suppose that $\Lambda_i > 0$, which implies $B_i = \frac{1}{1 - \delta_i} [J_i^n(X^d_t) - J_i^n(X^n_t)]$. In other words, retailer $i$’s profit is larger if the retailers order $X^d_t$, and she receives a positive bonus to compensate for ordering $X^n_t$. Then,

$$B_i + \frac{1}{1 - \delta_i} J_i^n(X^n_t) = \frac{1}{1 - \delta_i} [J_i^n(X^d_t) - J_i^n(X^n_t)] + \frac{1}{1 - \delta_i} J_i^n(X^n_t) = \frac{1}{1 - \delta_i} J_i^n(X^d_t),$$

and hence $i$ is not better off if she does not adopt the contract.

Now, suppose that $\Lambda_i \leq 0$—that is, retailer $i$’s profit is larger if the retailers order $X^n_t$ and she gives a side payment to other retailers to induce their acceptance of the contract. Observe that $J_i^n(X^n_t) \geq J_i^n(X^d_t)$, which implies $\sum_i \Lambda_i \leq 0$. This further means that $0 \leq \sum_{K^+} \Lambda_j \leq \sum_{K^-} (-\Lambda_j)$ and

$$0 \leq \frac{\sum_{K^+} (-\Lambda_j)}{\sum_{K^-} \Lambda_j} \leq 1. \quad \text{(A10)}$$

Now,

$$B_i + \frac{1}{1 - \delta_i} J_i^n(X^n_t) = \frac{1}{1 - \delta_i} [J_i^n(X^d_t) - J_i^n(X^n_t)] + \frac{\sum_{K^+} (-\Lambda_j)}{\sum_{K^-} \Lambda_j} \times \frac{1}{1 - \delta_i} J_i^n(X^n_t)$$

$$\geq \frac{1}{1 - \delta_i} [J_i^n(X^d_t) - J_i^n(X^n_t)] + \frac{1}{1 - \delta_i} J_i^n(X^n_t) = \frac{1}{1 - \delta_i} J_i^n(X^d_t),$$

where the inequality follows from (A10). Thus, the participation constraint is satisfied for all $i$.

**Early Adoption Constraint:** If the contract is adopted in period $t = 2$ instead of in period $t = 1$, the retailers order $X^d_t$ in period 1, and retailer $i$ realizes the payoff

$$J_i^n(X^d_t) + \delta_i B_i + \sum_{t=2}^{\infty} \delta_t^{t-1} J_i^n(X^n_t) = J_i^n(X^d_t) + \delta_i B_i + \frac{\delta_i}{1 - \delta_i} J_i^n(X^n_t).$$

The early adoption constraint holds if

$$B_i + \frac{1}{1 - \delta_i} J_i^n(X^n_t) \geq J_i^n(X^d_t) + \delta_i B_i + \frac{\delta_i}{1 - \delta_i} J_i^n(X^n_t).$$

First, suppose that $\Lambda_i > 0$, which implies $B_i = \frac{1}{1 - \delta_i} [J_i^n(X^d_t) - J_i^n(X^n_t)]$. Then,

$$J_i^n(X^d_t) + \delta_i B_i + \frac{\delta_i}{1 - \delta_i} J_i^n(X^n_t) = J_i^n(X^d_t) + \frac{\delta_i}{1 - \delta_i} [J_i^n(X^d_t) - J_i^n(X^n_t)] + \frac{\delta_i}{1 - \delta_i} J_i^n(X^n_t) = \frac{1}{1 - \delta_i} J_i^n(X^d_t),$$
Hence, retailer \( i \) does not benefit from late adoption of the contract.

Next, when \( \Lambda_i \leq 0 \), then \( J_i^p(X^d) - J_i^p(X^n) \leq 0 \), and (A10) implies

\[
B_i + \frac{1}{1-\delta_i} J_i^n(X^u) = \frac{1}{1-\delta_i} [J_i^n(X^d) - J_i^n(X^u)] + \frac{1}{1-\delta_i} J_i^n(X^u) = \frac{1}{1-\delta_i} J_i^n(X^d).
\]

Thus, retailer \( i \) prefers to adopt the contract in the first period.

CONTINUATION CONSTRAINT: We now want to show that a retailer never benefits from defecting. Recall that \( Z_t \) denotes the coalition structure in period \( t \), and suppose that retailer \( i \) orders a quantity different from \( X_i^Z_t \) and/or withhold some of her residuals. As a result, she pays a penalty, \( d_{it} \), in period \( t \), and is excluded from inventory sharing in all subsequent periods. We denote, with slight abuse of notation, \( X_i = X_i^Z_t \), and suppose that retailer \( i \)’s discounted payoff starting from period \( t \) is given by

\[
J_i^{Z_t}(X_t, H_t, E_t) + d_{it}(X_t, H_t, E_t) + \frac{\delta_i}{1-\delta_i} J_i^1(X_i^1).
\]

The continuation constraint holds if

\[
J_i^{Z_t}(X_t, H_t, E_t) + d_{it}(X_t, H_t, E_t) + \frac{\delta_i}{1-\delta_i} J_i^1(X_i^1) \leq \frac{1}{1-\delta_i} J_i^{Z_t}(X^Z_t).
\]

If \( i \in I^- \), then \( \Delta_{it} \leq 0 \), and \( d_{it} = \frac{1}{1-\delta_i} \left[ J_i^{Z_t}(X^Z_t) - \delta_i J_i^1(X_i^1) \right] - J_i^{Z_t}(X_t, H_t, E_t). \) Thus, \( i \) receives a payoff

\[
J_i^{Z_t}(X_t, H_t, E_t) + \frac{1}{1-\delta_i} \left[ J_i^{Z_t}(X^Z_t) - \delta_i J_i^1(X_i^1) \right] - J_i^{Z_t}(X_t, H_t, E_t) + \frac{\delta_i}{1-\delta_i} J_i^1(X_i^1) = \frac{1}{1-\delta_i} J_i^{Z_t}(X^Z_t),
\]

and \( i \) does not benefit from defection.

Now, suppose \( i \in I^+ \), and consequently \( \Delta_{it} > 0 \). This implies

\[
\frac{1}{1-\delta_i} \left[ J_i^{Z_t}(X^Z_t) - \delta_i J_i^1(X_i^1) \right] - J_i^{Z_t}(X_t, H_t, E_t) \geq 0.
\] (A11)

Notice that

\[
\sum_i \Delta_{it} = \sum_i \left\{ \frac{1}{1-\delta_i} \left[ J_i^{Z_t}(X^Z_t) - \delta_i J_i^1(X_i^1) \right] - J_i^{Z_t}(X_t, H_t, E_t) \right\}
\]

\[
= \frac{\delta_i}{1-\delta_i} \left\{ J_i^{Z_t}(X^Z_t) - J_i^1(X_i^1) \right\} + J_i^{Z_t}(X^Z_t) - J_i^{Z_t}(X_t, H_t, E_t) \geq 0
\]

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where the inequality holds because $X^{Z_t}$ with complete residual sharing maximizes the system profit when the state is $Z_t$ and systems with inventory-sharing retailers generate higher profit than systems without inventory sharing. As a result, $\sum_{I^+_i} \Delta_{jt} \geq \sum_{I^-_i} (-\Delta_{jt})$, and

$$0 \leq \frac{\sum_{I^-_i} (-\Delta_{jt})}{\sum_{I^+_i} \Delta_{jt}} \leq 1.$$  \hspace{1cm} (A12)

Thus, retailer $i$ receives a payoff

$$J^Z_i(X_t, H_t, E_t) + \left\{ \frac{1}{1-\delta_i} \left[ J^Z_i(X^{Z_t}) - \delta_i J^1_i(X^j_t) \right] - J^Z_i(X_t, H_t, E_t) \right\} \times \frac{\sum_{I^-_i} (-\Delta_{jt})}{\sum_{I^+_i} \Delta_{jt}} + \frac{\delta_i}{1-\delta_i} J^1_i(X^j_t)$$

$$\leq J^Z_i(X_t, H_t, E_t) + \frac{1}{1-\delta_i} \left[ J^Z_i(X^{Z_t}) - \delta_i J^1_i(X^j_t) \right] - J^Z_i(X_t, H_t, E_t) + \frac{\delta_i}{1-\delta_i} J^1_i(X^j_t) = \frac{1}{1-\delta_i} J^Z_i(X^{Z_t}),$$

where the inequality follows from (A11) and (A12). As a result, $i$ prefers not to defect in any period. \hfill \Box

**Appendix B - Conditions for achieving a first-best outcome in a repeated game**

In this Appendix, we provide some additional interpretation of condition (2). Suppose that retailer $i$ orders quantity $X_i$. After demand $D_i$ is realized, she may be left with residual demand, $\bar{E}_i$, or with residual supply, $\bar{H}_i$. Suppose that all retailers share their entire residuals, $H_j = \bar{H}_j, E_j = \bar{E}_j, \forall j$. Given $H_{-i} = (H_1, H_2, \ldots, H_{i-1}, H_{i+1}, \ldots, H_n)$ and $E_{-i} = (E_1, E_2, \ldots, E_{i-1}, E_{i+1}, \ldots, E_n)$, dual prices $\lambda_j$ and $\mu_j$ will have different values for different $E_i$ and $H_i$. Note, however, that the values of dual prices change in the form of a step function. $\lambda_j$, the price for residual inventory, is non-decreasing with $E_i$ for $j \neq i$. Similarly, $\mu_j$ is non-decreasing with $H_i$. Thus, there are a finite number of jumps for both $\lambda_j$ and $\mu_j$. In other words, we can find values

$$e_m \in (-X_i, \infty), e_m < e_{m+1}, m = 1, 2, \ldots \text{ and } h_l \in (-\infty, X_i), h_l < h_{l+1}, l = 1, 2, \ldots$$

such that given $H_{-i}$ and $E_{-i}$, $\lambda_j$ does not change for any $E_i \in (e_m, e_{m+1})$ (that is, when $D_i \in (X_i + e_m, X_i + e_{m+1})$), and $\mu_j$ does not change for any $H_i \in (h_l, h_{l+1})$ (that is, when $D_i \in (X_i - h_l, X_i - h_{l+1})$). At the same time,

$$\lambda_j \left( E_i \in (e_l, e_{l+1}), H_{-i}, E_{-i} \right) \neq \lambda_j \left( E_i \in (e_{l+1}, e_{l+2}), H_{-i}, E_{-i} \right) \text{ and }$$

$$\mu_j \left( H_i \in (h_l, h_{l+1}), H_{-i}, E_{-i} \right) \neq \mu_j \left( H_i \in (h_{l+1}, h_{l+2}), H_{-i}, E_{-i} \right).$$

Thus, we can define

$$\lambda_j^{m} \left( H_{-i}, E_{-i} \right) = \lambda_j \left( E_i \in (e_m, e_{m+1}), H_{-i}, E_{-i} \right), \quad \mu_j^{l} \left( H_{-i}, E_{-i} \right) = \mu_j \left( H_i \in (h_l, h_{l+1}), H_{-i}, E_{-i} \right).$$
With some abuse of notation, we write $\lambda^m_j$ and $\mu^l_j$ when it is clear what values of $(H_{-i}, E_{-i})$ they refer to. We can now define the total variation of dual allocations for dual prices $\lambda_j$, $\mu_j$ w.r.t. retailer $i$’s ordering quantity $X_i$ given the ex post residuals of retailers other than $i$:

$$TV^{(i)}_{\lambda_j}(X_i, H_{-i}, E_{-i}) = H_j \sum_{m} f_i(X_i + e_m) \left[ \lambda^{m+1}_j(H_{-i}, E_{-i}) - \lambda^m_j(H_{-i}, E_{-i}) \right]$$

$$TV^{(i)}_{\mu_j}(X_i, H_{-i}, E_{-i}) = E_j \sum_{l} f_i(X_i - h_l) \left[ \mu^{l+1}_j(H_{-i}, E_{-i}) - \mu^l_j(H_{-i}, E_{-i}) \right].$$

The total variation of dual allocations for dual prices $\lambda_j$ and $\mu_j$ is, therefore, the expected “jump amount” of the allocations resulting from residual supply and residual demand, respectively. We now take the expectation over $H_{-i}, E_{-i}$ at $X_{-i} = X^n_{-i}$ and denote

$$ETV^{(i)}_{\lambda_j}(X_i) = E[TV^{(i)}_{\lambda_j}(X_i, H_{-i}, E_{-i})],$$

$$ETV^{(i)}_{\mu_j}(X_i) = E[TV^{(i)}_{\mu_j}(X_i, H_{-i}, E_{-i})].$$

We use these expressions to describe the retailers who are “alike” in a more general sense, as follows. In an inventory-sharing game with $n$ retailers, the retailers are relaxed-symmetric if, for any $i$, the sums of the expected total variation w.r.t. $i$ for all retailers other than $i$ are equal for both dual prices if $i$ orders a system-optimal stocking quantity:

$$\sum_{j \neq i} ETV^{(i)}_{\lambda_j}(X^n_i) = \sum_{j \neq i} ETV^{(i)}_{\mu_j}(X^n_i) \quad \forall i, j. \quad (B1)$$

Observe that, when $n = 2$, (B1) corresponds to the sufficient and necessary condition for achieving a first-best solution (see N&S 2008). This relationship continues to hold when we have an arbitrary number of retailers: our next result follows from Theorem B1 after observing that (B1) corresponds to (2).

**Theorem B1.** In an inventory-sharing game with $n$ relaxed-symmetric retailers, if $J^n(X)$ is unimodal in $X$, a first-best solution can be induced with dual allocations when $\delta > \delta^*_n$.  

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