Abstract

In this paper we study the problem of recovering an low-rank positive semidefinite matrix from linear measurements. Our algorithm, which we call Procrustes Flow, starts from an initial estimate obtained by a thresholding scheme followed by gradient descent on a non-convex objective. We show that as long as the measurements obey a standard restricted isometry property, our algorithm converges to the unknown matrix at a geometric rate. In the case of Gaussian measurements, such convergence occurs for a $n \times n$ matrix of rank $r$ when the number of measurements is $\Omega(nr)$.

1 Introduction

In numerous applications from signal processing, machine learning, control, and wireless communications, it is often of interest to find a positive semidefinite matrix of minimal rank obeying a set of linear equations. In particular, one wishes to solve problems of the form

$$\min_{X \in \mathbb{R}^{n \times n}} \text{rank}(X) \quad \text{s.t.} \quad A(X) = b \quad \text{and} \quad X \succeq 0,$$

where $A : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^m$ is a known affine transformation that maps matrices to vectors. More specifically, the $k$-th entry of $A(X)$ is $\langle A_k, X \rangle := \text{Tr}(A_k X)$, where each $A_k \in \mathbb{R}^{n \times n}$ is symmetric.

Since the early seventies, a very popular heuristic for solving such problems has been to replace $X$ with a low-rank factorization $X = UU^T$ and solve matrix quadratic equations of the form

$$\text{find} \quad U \in \mathbb{R}^{n \times r} \quad \text{s.t.} \quad A(U U^T) = b,$$

via a local search heuristic [Ruh74]. Many researchers have demonstrated that such heuristics work well in practice for a variety of problems [RS05, Fun06, LRS+10, RR13]. However, these procedures lack strong guarantees associated with convex programming heuristics for solving (1.1).

In this paper we show that a local search heuristic provably solves (1.2) under standard restricted isometry assumptions on the linear map $A$. For standard ensembles of equality constraints, we demonstrate that $X$ can be estimated to precision $\epsilon$ by such local search as long as we have $\Omega(nr)$.
equations. 1 This is merely a constant factor more than the number of parameters needed to specify a $n \times n$ rank $r$ matrix. Specialized to a random Gaussian model, our work improves upon recent and independent work by Zheng and Lafferty [ZL15].

2 Algorithm: Procrustes Flow

In this paper we study a local search heuristic for solving matrix quadratic equations of the form (1.2) which consists of two components: (1) a careful initialization obtained by a projected gradient scheme on $n \times n$ matrices, and (2) a series of successive refinements of this initial solution via a gradient descent scheme. This algorithm is a natural extension of the Wirtinger Flow algorithm developed in [CLS14b] for solving vector quadratic equations. Following [CLS14b], we shall refer to the combination of these two steps as the Procrustes Flow (PF) algorithm detailed in Algorithm 1.

2.1 Initialization via low-rank projected gradients

In the initial phase of our algorithm we start from $Z_0 = 0_{n \times n}$ and apply successive updates of the form on rank $r$ matrices of size $n \times n$

$$Z_{\tau+1} = P_r \left( Z_\tau - \alpha_{\tau+1} \sum_{k=1}^{m} (\langle A_k, Z_\tau \rangle - b_k) A_k \right),$$

(2.1)

for $T_{\text{init}}$ iterations. Here, $P_r$ denotes projection onto rank-$r$ positive semidefinite (PSD) matrices which can be computed efficiently via Lanczos methods. We set our initialization to an $n \times r$ matrix $U_0$ obeying $Z_{\text{init}} = U_0 U_0^T$.

Updates of the form (2.1) have a long history in compressed sensing/matrix sensing literature (see e.g. [TG07, GK09, NT09, NV09, BD09, MJD09, CCS10]). Furthermore, using the first step of the update (2.1) for the purposes of initialization has also been proposed in previous work (see e.g. [AM07, KMO10, JNS12]).

2.2 Successive refinement via gradient descent

As mentioned earlier, we are interested in finding a matrix $U \in \mathbb{R}^{n \times r}$ obeying matrix quadratic equations of the form $b = A(UU^T)$. We wish to refine our initial estimate by solving the following non-convex optimization problem

$$\min_{U \in \mathbb{R}^{n \times r}} f(U) := \frac{1}{4} \| A(UU^T) - b \|_2^2 = \frac{1}{4} \sum_{k=1}^{m} (\langle A_k, UU^T \rangle - b_k)^2,$$

(2.2)

which minimizes the misfit in our quadratic equations via the square loss. To solve (2.2), starting from our initial estimate $U_0 \in \mathbb{R}^{n \times r}$ we apply the successive updates

$$U_{\tau+1} := U_{\tau} - \mu_{\tau+1} \frac{\sigma_2^2(U_0)}{\sigma_1(U_0)} \nabla f(U_{\tau}) = U_{\tau} - \mu_{\tau+1} \frac{\sigma_2^2(U_0)}{\sigma_1(U_0)} \left( \sum_{k=1}^{m} (\langle A_k, U_{\tau} U_{\tau}^T \rangle - b_k) A_k U_{\tau} \right).$$

(2.3)

Here and throughout, for a matrix $X$, $\sigma_\ell(X)$ denotes the $\ell$-th largest singular value of $X$. We note that the update (2.3) is essentially gradient descent with a carefully chosen step size.

1Here and throughout we use $f(x) = \Omega(g(x))$ if there is a positive constant $C$ such that $f(x) \geq C g(x)$ for all $x$ sufficiently large.
Algorithm 1 Procrustes Flow (PF)

Require: \( \{A_k\}_{k=1}^m, \{b_k\}_{k=1}^m, \{\alpha_\tau\}_{\tau=1}^\infty, \{\mu_\tau\}_{\tau=1}^\infty, T_{\text{init}} \in \mathbb{N}. \)

// Initialization phase
\( Z_0 := 0_{n \times n}. \)

repeat
  \( Z_{\tau+1} \leftarrow P_r(Z_\tau - \alpha_\tau \sum_{k=1}^m (\langle A_k, Z_\tau \rangle - b_k)A_k), \)
  // Projection onto rank \( r \) PSD matrices.
until \( \tau = T_{\text{init}} \)

\( Q \Sigma Q^T := Z_{T_{\text{init}}}, \)
  // SVD of \( Z_{T_{\text{init}}}, \) with \( Q \in \mathbb{R}^{n \times r}, \Sigma \in \mathbb{R}^{r \times r}. \)
\( U_0 := Q \Sigma^{1/2}. \)

// Gradient descent phase
repeat
  \( U_{\tau+1} \leftarrow U_\tau - \mu_{\tau+1} \frac{\sigma_2(U_0)}{\sum_{k=1}^m (\langle A_k, U_\tau U_\tau^T \rangle - b_k)A_kU_\tau), \)
until convergence

3 Main Results

For our theoretical results we shall focus on affine maps \( A \) which are isotropic and obey the matrix Restricted Isometry Property (RIP).

Definition 3.1 (Isotropic mapping). We say the random map \( A \) is an isotropic mapping if
\[
\mathbb{E}[\|A(X)\|_F^2] = \|X\|_F^2,
\]
holds for any fixed \( X \in \mathbb{R}^{n \times n} \) which is independent of \( A \).

Definition 3.2 (Restricted Isometry Property (RIP) [CT05, RFP10]). The map \( A \) satisfies \( r\text{-RIP} \) with constant \( \delta_r \), if
\[
(1 - \delta_r) \|X\|_F^2 \leq \|A(X)\|_F^2 \leq (1 + \delta_r) \|X\|_F^2,
\]
holds for all matrices \( X \in \mathbb{R}^{n \times n} \) of rank at most \( r \).

We are interested in finding a matrix \( U \in \mathbb{R}^{n \times r} \) obeying quadratic equations of the form
\[
A(UU^T) = b, \tag{3.1}
\]
where we assume \( b = A(XX^T) \) for a planted solution \( X \in \mathbb{R}^{n \times r} \). We wish to understand when the Procrustes Flow algorithm recovers this planted solution \( X \). We note that this is only possible up to a certain rotational factor as if \( U \) obeys (3.1), then so does any matrix \( UR \) with \( R \in \mathbb{R}^{r \times r} \) an orthonormal matrix satisfying \( R^T R = I_r \). This naturally leads to defining the distance between two matrices \( U, X \in \mathbb{R}^{n \times r} \) as
\[
\text{dist}(U, X) := \min_{R \in \mathbb{R}^{r \times r}; R^T R = I_r} \|U - XR\|_F. \tag{3.2}
\]
We note that this distance is the solution to the classic orthogonal Procrustes problem (hence the name of the algorithm). It is known that the optimal rotation matrix \( R \) minimizing \( \|U - XR\|_F \) is equal to \( R = AB^T \), where \( A\Sigma B^T \) is the singular value decomposition (SVD) of \( X^TU \). We now have all of the elements in place to state our main result.
Theorem 3.3. Let $X \in \mathbb{R}^{n \times r}$ be an arbitrary matrix with singular values $\sigma_1(X) \geq \sigma_2(X) \geq \ldots \geq \sigma_r(X) > 0$ and condition number $\kappa = \sigma_1(X)/\sigma_r(X)$. Also, let $b = AXX^T \in \mathbb{R}^m$ be $m$ quadratic samples. Furthermore, assume the mapping $A$ is isotropic and obeys rank-$4r$ RIP with RIP constant $\delta_{4r} \leq 1/10$. Then, using $T_{\text{init}} \geq \log(\sqrt{r}\kappa^2) + 2$ iterations of the initialization phase of Procrustes Flow with $\alpha_r = 1/m$ yields a solution $U_0$ obeying
\begin{equation}
\text{dist} (U_0, X) \leq \frac{1}{4} \sigma_r(X).
\end{equation}
Furthermore, take a constant step size $\mu_\tau = \mu$ for all $\tau = 1, 2, \ldots$ and assume $\mu \leq 4/75$. Then, starting from any initial solution obeying (3.3), we have
\begin{equation}
\text{dist} (U_\tau, X) \leq \frac{1}{4} \left( 1 - \frac{24}{125} \frac{\mu}{\kappa^4} \right)^{\tau/2} \sigma_r(X).
\end{equation}

The above theorem shows that Procrustes Flow algorithm achieves a good initialization under the isotropy and RIP assumptions on the mapping $A$. Also, starting from any sufficiently accurate initialization the algorithm exhibits geometric convergence to the unknown matrix $X$. We note that in the above result we have not attempted to optimize the constants. Furthermore, there is a natural tradeoff involved between the upper bound on the RIP constant, the radius in which PF is contractive (3.3), and its rate of convergence (3.4). In particular, as it will become clear in the proofs one can increase the radius in which PF is contractive (increase the constant $1/4$ in (3.3)) and the rate of convergence (increase the constant $24/125$ in (3.4)) by assuming a smaller upper bound on the RIP constant.

The most common measurement ensemble which satisfies the isotropy and RIP assumptions is the spiked Gaussian ensemble, where each symmetric matrix $A_k$ has $N(0,1/m)$ entries on the diagonal and $N(0,1/2m)$ entries elsewhere. For this ensemble to achieve a RIP constant of $\delta_r$, we require at least $m = \Omega(\frac{1}{\sqrt{r}}nr)$ measurements. Applying Theorem 3.3 to this measurement ensemble, we conclude that the Procrustes Flow algorithm yields a solution with relative error $\epsilon$ ($\text{dist}(\hat{X}, X)/\|X\|_F \leq \epsilon$) in $\mathcal{O}(\log(\sqrt{r}/\epsilon))$ iterations using only $\Omega(nr)$ measurements. We would like to note that if more measurements are available it is not necessary to use multiple projected gradient updates in the initialization phase. In particular, for the Gaussian model if $m = \Omega(nr^2\kappa^4)$, then (3.3) will hold after the first iteration ($T_{\text{init}} = 1$).

How to verify the initialization is complete. Theorem 3.3 requires that $T_{\text{init}} = \Omega(\log(\sqrt{r}\kappa^2))$, but $\kappa$ is a property of $X$ and is hence unknown. However, as long as the RIP constant $\delta_{2r}$ satisfies $\delta_{2r} \leq 1/10$, then we can use each iterate of initialization to test whether or not we have entered the radius of convergence. The following lemma establishes a sufficient condition we can check using only information from $Z_\tau$. The proof is deferred to Appendix B.

Lemma 3.4. Assume the RIP constant of $A$ satisfies $\delta_{2r} \leq 1/10$. Let $Z_\tau$ denote the $\tau$-th step of initialization in Procrustes Flow, and let $U_0 \in \mathbb{R}^{n \times r}$ be the such that $Z_\tau = U_0 U_0^T$. Define
\begin{equation}
\epsilon_\tau := \|A(Z_\tau) - b\|_{\ell_2} = \|A(Z_\tau - XX^T)\|_{\ell_2}
\end{equation}
Then, if the following condition holds
\begin{equation}
\epsilon_\tau \leq \frac{3}{20} \sigma_r(Z_\tau),
\end{equation}
we have that

\[ \text{dist}(U_0, X) \leq \frac{1}{4} \sigma_r(X). \]

One might consider using solely the projected gradient updates (i.e. set \( T_{\text{init}} = \infty \)) as in previous approaches [TG07, GK09, NT09, NV09, BD09, MJD09, CCS10]. We note that the projected gradient updates in the initialization phase require computing the first \( r \) eigenvectors of a PSD matrix whereas the gradient updates do not require any eigenvector computations. Such eigenvector computations may be prohibitive compared to the gradient updates, especially when \( n \) is large and for ensembles where matrix-vector multiplication is fast. We would like to emphasize, however, that for small \( n \) and dense matrices using projected gradient updates may be more efficient due to faster convergence rate. Our scheme is a natural interpolation: one could only do projected gradient steps, or one could do one projected gradient step. Here we argue that very few projected gradients provide sufficient initialization such that gradient descent converges geometrically.

4 Related work

There is a vast literature dedicated to low-rank matrix recovery/sensing and semi-definite programming. We shall only focus on the papers most related to our framework.

Recht, Fazel, and Parrilo were the first to study low-rank solutions of linear matrix equations under RIP assumptions [RFP10]. They showed that if the rank-\( r \) RIP constant of \( A \) is less than a fixed numerical constant, then the matrix with minimum trace satisfying the equality constraints coincided with the minimum rank solution. In particular, for the Gaussian ensemble the required number of measurements is \( \Omega(nr) \) [CP11]. Subsequently, a series of papers [CR09, Gro11, Rec11, CLS14a] showed that trace minimization and related convex optimization approaches also work for other measurement ensembles such as those arising in matrix completion and related problems. In this paper we have established a similar result to [RFP10] (albeit only for PSD matrices). We require the same order of measurements \( \Omega(nr) \) but use a more computationally friendly local search algorithm. Also related to this work are projection gradient schemes with hard thresholding [TG07, GK09, NT09, NV09, BD09, MJD09, CCS10]. Such algorithms enjoy similar guarantees to that of [RFP10] and this work. Indeed, we utilize such results in the initialization phase of our algorithm. However, such algorithms require a rank-\( r \) SVD in each iteration which may be expensive for large problem sizes. We would like to emphasize, however, that for small problem sizes and dense matrices (such as Gaussian ensembles) such algorithms may be faster than gradient descent approaches such as ours.

More recently, there have been a few results using non-convex optimization schemes for matrix recovery problems. In particular, theoretical guarantees for matrix completion have been established using manifold optimization\(^2\) [KMO10] and alternating minimization [Kes12] (albeit with the caveat of requiring a fresh set of samples in each iteration). Later on, Jain et.al. [JNS12] analyzed the performance of alternating minimization under similar modeling assumptions to [RFP10] and this paper. However, the requirements on the RIP constant in [JNS12] are more stringent compared to [RFP10] and ours. In particular, the authors require \( \delta_{4r} \leq c/r \) whereas we only require \( \delta_{4r} \leq c \). We should note, however, that the authors study general low-rank matrices rather than PSD ones.

\(^2\)See [JBAS10] for related schemes using manifold optimization.
studied in this paper (we defer the analysis of general low-rank matrices to a future version of this paper). Specialized to the Gaussian model, the results of [JNS12] require $\Omega(nr^3\kappa^4)$ measurements.\(^3\)

Our algorithm and analysis are inspired by the recent paper [CLS14b] by Candes, Li and Soltanolkotabi. See also [Sol14, CLM15] for some stability results. In [CLS14b] the authors introduced a local regularity condition to analyze the convergence of a gradient descent-like scheme for phase retrieval. We use a similar regularity condition but generalize it to ranks higher than one. Recently, independent of our work, Zheng and Lafferty [ZL15] provided an analysis of gradient descent using (2.2) via the same regularity condition. Zheng and Lafferty focus on the Gaussian ensemble, and establish a sample complexity of $m = \Omega(nr^2\kappa^4\log n)$.\(^4\) In comparison we only require $\Omega(nr^3)$ measurements removing both the dependence on $\kappa$ in the sample complexity and improving the asymptotic rate. We would like to emphasize that the improvement in our result is not just due to the more sophisticated initialization scheme. In particular, Zheng and Lafferty show geometric convergence (albeit at a slower rate) starting from any initial solution obeying $\text{dist}(U_0, X) \leq c\sigma_r(X)$ as long as the number of measurements obeys $m = \Omega(nr\kappa^4\log n)$. In contrast, we establish geometric convergence starting from the same neighborhood of $U_0$ with only $\Omega(nr^3)$ measurements.

Moreover, the theory of restricted isometries in our work considerably simplifies the analysis. Finally, we would also like to mention [SOR14] for guarantees using stochastic gradient algorithms. The results of [SOR14] are applicable to a variety of models; focusing on the Gaussian ensemble, the authors require $\Omega((nr\log n)/\epsilon)$ samples to reach a relative error of $\epsilon$. In contrast, our sample complexity is independent of the desired relative error $\epsilon$. However, their algorithm only requires a random initialization.

5 Proofs

Before we dive into the details of the proofs, we would like to mention that we will prove our results using the update

$$U_{t+1} = U_t - \frac{\mu}{\kappa^2 \|X\|^2} \nabla f(U_t),$$

in lieu of the PF update

$$U_{t+1} = U_t - \mu_{\text{PF}} \frac{\sigma_2^2(U_0)}{\|U_0\|^4} \nabla f(U_t).$$

As we prove in Section 5.3, our initial solution obeys $\|U_0U_0^T - XX^T\| \leq \sigma_r^2(X)/4$. Hence, applying Weyl’s inequality we can conclude that

$$\frac{\sigma_r^2(U_0)}{\sigma_1^2(U_0)} \leq \frac{\sigma_r^2(X) + \sigma_r^2(X)/4}{(\sigma_r^2(X) - \sigma_1^2(X))/4}^2 \leq 3\frac{\sigma_r^2(X)}{\sigma_1^2(X)},$$

\(^3\)The authors also propose a stage-wise algorithm with improved sample complexity of $\Omega(nr^3\tilde{\kappa}^4)$ where $\tilde{\kappa}$ is a local condition number defined as the ratio of the maximum ratio of two successive eigenvalues. We note, however, that in general $\tilde{\kappa}$ can be as large as $\kappa$.

\(^4\)We note that the paper [ZL15] defines $\kappa$ in terms of the larger matrix $XX^T$, whereas our convention is to define $\kappa$ for the smaller matrix $X$. 
and similarly,
\[
\frac{\sigma_r^2(U_0)}{\sigma_r^4(U_0)} \geq \frac{12 \sigma_r^2(X)}{25 \sigma_1^4(X)}.
\] (5.4)

Thus, any result proven for the update (5.1) will automatically carry over to the PF update with a simple rescaling of the upper bound on the step size via (5.3). Furthermore, we can upper bound the convergence rate of gradient descent using the PF update in terms of properties of \(X\) instead of \(U_0\) via (5.4).

5.1 Preliminaries

We start with a well known characterization of RIP.

**Lemma 5.1.** [Can08] Let \(A\) satisfy 4r-RIP with constant \(\delta_{4r}\). Then, for all fixed matrices \(X, Y\) of rank at most \(2r\) which are independent of \(A\), we have
\[
|\langle A(X), A(Y) \rangle - \langle X, Y \rangle| \leq \delta_{4r} \|X\|_F \|Y\|_F.
\]

Next, we state a recent result which characterizes the convergence rate of projected gradient descent onto general non-convex sets specialized to our problem. See [MJD09] for related results using singular value hard thresholding.

**Lemma 5.2.** [ORS15] Let \(X \in \mathbb{R}^{n \times r}\) be an arbitrary matrix. Also, let \(b = A(XX^T) \in \mathbb{R}^m\) be \(m\) quadratic samples. Consider the iterative updates
\[
Z_{\tau+1} \leftarrow \mathcal{P}_r \left( Z_\tau - \frac{1}{m} \sum_{k=1}^m (\langle A_k, Z_\tau \rangle - b_k) Z_\tau \right),
\]
where \(\mathcal{P}_r\) projects its input matrix onto the rank-\(r\) PSD cone. Then,
\[
\|Z_\tau - XX^T\|_F \leq \rho(A)^\tau \|Z_0 - XX^T\|_F,
\]
holds. Here, \(\rho(A)\) is defined as
\[
\rho(A) := 2 \sup_{\|X\|_F = 1, \text{rank}(X) \leq 2, \|Y\|_F = 1, \text{rank}(Y) \leq 2} |\langle A(X), A(Y) \rangle - \langle X, Y \rangle|.
\]

We shall make repeated use of the following lemma which upper bounds \(\|UU^T - XX^T\|_F\) by some factor of \(\text{dist}(U, X)\).

**Lemma 5.3.** For any \(U \in \mathbb{R}^{n \times r}\) obeying \(\text{dist}(U, X) \leq \frac{1}{4} \|X\|\), we have
\[
\|UU^T - XX^T\|_F \leq \frac{9}{4} \|X\| \|U - XR\|_F.
\]

**Proof.**
\[
\|UU^T - XX^T\|_F = \|U(U - XR)^T + (U - XR)(XR)^T\|_F \\
\leq (\|U\| + \|X\|) \|U - XR\|_F \\
\leq \frac{9}{4} \|X\| \|U - XR\|_F.
\]
Finally, we also need the following lemma which upper bounds dist($U, X$) by some factor of $\|UU^T - XX^T\|_F$. We defer the proof of this result to Appendix A.

Lemma 5.4. For any $U, X \in \mathbb{R}^{n \times r}$, we have

$$\text{dist}(U, X)^2 \leq \frac{1}{2(\sqrt{2} - 1)\sigma_r^2(X)} \left\|UU^T - XX^T\right\|^2_F.$$

5.2 Proof of convergence of gradient descent updates (Equation (3.4))

We first outline the general proof strategy. Please see [CLS14b, Sections 2.3 and 7.9] for related arguments. The first step is to show that gradient descent on the expected value of the function, which we call $F(U) := \mathbb{E}[f(U)]$, exhibits geometric convergence in a small neighborhood around $X$. The standard approach in optimization to show this is to prove that the function exhibits strong convexity. However, it is not possible for $F(U)$ to be strongly convex in any neighborhood around $X$.\(^5\) Thus, we rely on the approach used by [CLS14b], which establishes a sufficient condition that only relies on first-order information along certain trajectories. After showing the sufficient condition holds in expectation, we use standard RIP results to show that this condition also holds for the function $f(U)$ with high probability.

To begin our analysis, we start with the following formulas for the gradient of $f(U)$ and $F(U)$

$$\nabla f(U) = \sum_{k=1}^m \langle A_k, UU^T - XX^T \rangle A_k U, \quad \nabla F(U) = (UU^T - XX^T)U.$$

Throughout the proof $R$ is the solution to the orthogonal Procrustes problem. That is,

$$R = \arg \min_{\tilde{R} \in \mathbb{R}^{n \times n} : \tilde{R}^T \tilde{R} = I_r} \left\|U - X\tilde{R}\right\|_F,$$

with the dependence on $U$ omitted for sake of exposition. The following definition defines a notion of strong convexity along certain trajectories of the function.

Definition 5.5. (Regularity condition, [CLS14b]) Let $X \in \mathbb{R}^{n \times r}$ be a global optimum of a function $f$. Define the set $B(\delta)$ as

$$B(\delta) := \{U \in \mathbb{R}^{n \times r} : \text{dist}(U, X) \leq \delta\}.$$

The function $f$ satisfies a regularity condition, denoted by $RC(\alpha, \beta, \delta)$, if for all matrices $U \in B(\delta)$ the following inequality holds:

$$\langle \nabla f(U), U - XR \rangle \geq \frac{1}{\alpha} \left\|U - XR\right\|^2_T + \frac{1}{\beta} \left\|\nabla f(U)\right\|^2_T.$$

If a function satisfies $RC(\alpha, \beta, \delta)$, then as long as gradient descent starts from a point $U_0 \in B(\delta)$, it will have a geometric rate of convergence to the optimum $X$. This is formalized by the following lemma.

\(^5\)Such a proof is only possible in the special case when $r = 1$. 

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Lemma 5.6. [CLS14b] If $f$ satisfies $RC(\alpha, \beta, \delta)$ and $U_0 \in B(\delta)$, then the gradient descent update

$$U_{\tau + 1} \leftarrow U_{\tau} - \mu \nabla f(U_{\tau}),$$

with step size $0 < \mu \leq 2/\beta$ obeys $U_{\tau} \in B(\delta)$ and

$$\text{dist}^2(U_{\tau}, X) \leq \left(1 - \frac{2\mu}{\alpha}\right) \tau \text{dist}^2(U_0, X),$$

for all $\tau \geq 0$.

The proof is complete by showing that the regularity condition holds. To this end, we first show in Lemma 5.7 below that the expected function $F(U)$ satisfies a slightly stronger variant of the regularity condition from Definition 5.5. We then show in Lemma 5.8 that the gradient of $f$ is always close to the gradient of $F$.

Lemma 5.7. Let $F(U) = \|UU^T - XX^T\|_F$ denote the average function. For all $U$ obeying

$$\|U - XR\| \leq \frac{1}{4} \sigma_r(X),$$

we have

$$\langle \nabla F(U), U - XR \rangle - \frac{1}{20} \left(\|UU^T - XX^T\|_F^2 + \|U - XRU^T\|_F^2\right) \geq \frac{\sigma_r^2(X)}{4} \|U - XR\|_F^2 + \frac{4}{25} \frac{1}{\kappa^2} \|X\|^2 \|\nabla F(U)\|_F^2. \quad (5.5)$$

Lemma 5.8. Let $A$ be a linear map obeying rank-$4r$ RIP with constant $\delta_{4r}$. For any $V \in \mathbb{R}^{n \times r}$ and any $U \in \mathbb{R}^{n \times r}$ obeying $\text{dist}(U, X) \leq \frac{1}{4} \|X\|$ and independent of $A$, we have

$$\|\langle \nabla F(U) - \nabla f(U), V \rangle\| \leq \delta_{4r} \left\lVert UU^T - XX^T \right\rVert_F \left\lVert VU^T \right\rVert_F.$$

This immediately implies that for any $U \in \mathbb{R}^{n \times r}$ obeying $\text{dist}(U, X) \leq \frac{1}{4} \|X\|$ and independent of $A$, we have

$$\|\nabla f(U) - \nabla F(U)\|_F \leq \delta_{4r} \left\lVert UU^T - XX^T \right\rVert_F \|U\|.$$

We shall prove these two lemmas in Sections 5.2.1 and 5.2.2. However, we first explain how the regularity condition follows from these two lemmas. To begin, note that

$$\langle \nabla F(U), U - XR \rangle = \langle \nabla f(U), U - XR \rangle + \langle \nabla F(U) - \nabla f(U), U - XR \rangle$$

$$\leq (a) \langle \nabla f(U), U - XR \rangle + \frac{1}{10} \|UU^T - XX^T\|_F \left\lVert U - XRU^T \right\rVert_F$$

$$\leq (b) \langle \nabla f(U), U - XR \rangle + \frac{1}{20} \left(\|UU^T - XX^T\|_F^2 + \|U - XRU^T\|_F^2\right) \quad (5.6)$$

where (a) follows from Lemma 5.8 by using the fact that $\delta_{4r} \leq \frac{1}{10}$ as assumed in the statement of Theorem 3.3 and (b) follows from $2ab \leq a^2 + b^2$. 9
Combining (5.6) with Lemma 5.7 for any \( U \) obeying \( \| U - XR \| \leq \frac{1}{4} \sigma_r(X) \), we have

\[
\langle \nabla f(U), U - XR \rangle \geq \frac{\sigma_r^2(X)}{4} \| U - XR \|_F^2 + \frac{4}{25} \frac{1}{\kappa^2} \| X \|_2^2 \| \nabla F(U) \|_F^2.
\]  

(5.7)

Furthermore, for any \( U \) obeying \( \text{dist}(U, X) \leq \frac{1}{4} \| X \| \), we have

\[
\| \nabla F(U) \|_F^2 \geq \frac{1}{2} \| \nabla f(U) \|_F^2 - \| \nabla F(U) - \nabla f(U) \|_F^2.
\]

(5.8)

(b) \[ \geq \frac{1}{2} \| \nabla f(U) \|_F^2 - \delta^2_{4r} \| UU^T - XX^T \|_F^2 \]

(c) \[ \geq \frac{1}{2} \| \nabla f(U) \|_F^2 - \frac{25}{16} \delta^2_{4r} \| X \|_2^2 \| UU^T - XX^T \|_F^2 \]

(d) \[ \geq \frac{1}{2} \| \nabla f(U) \|_F^2 - 8 \delta^2_{4r} \| X \|_2^4 \| U - XR \|_F^2. \]

Here, (a) holds by the triangle inequality and the inequality \((a - b)^2 \geq a^2/2 - b^2\), (b) holds by Lemma 5.8, (c) follows from the fact that for \( \text{dist}(U, X) \leq \frac{1}{4} \| X \| \), we have \( \| U \| \leq \frac{5}{4} \| X \| \), and (d) follows from Lemma 5.3.

Equation (5.8) immediately implies

\[
\frac{4}{25} \frac{1}{\kappa^2} \| X \|_2^2 \| \nabla F(U) \|_F^2 \geq \frac{1}{2} \| \nabla f(U) \|_F^2 - \frac{25}{16} \delta^2_{4r} \| X \|_2^2 \| U - XR \|_F^2.
\]

(5.9)

Plugging the latter into (5.7), and assuming \( \delta_{4r} \leq \frac{1}{10} \), we conclude that for any \( \| U - XR \| \leq \frac{1}{4} \sigma_r(X) \) we have

\[
\langle \nabla f(U), U - XR \rangle \geq \left( \frac{1}{4} - \frac{32}{25} \delta^2_{4r} \right) \sigma_r^2(X) \| U - XR \|_F^2 + \frac{2}{25} \frac{1}{\kappa^2} \| X \|_2^2 \| \nabla f(U) \|_F^2
\]

\[
\geq \frac{1}{5} \sigma_r^2(X) \| U - XR \|_F^2 + \frac{2}{25} \frac{1}{\kappa^2} \| X \|_2^2 \| \nabla f(U) \|_F^2.
\]

Since \( \text{dist}(U, X) \leq \frac{1}{4} \sigma_r(X) \) implies \( \| U - XR \| \leq \frac{1}{4} \sigma_r(X) \), the last inequality shows that \( f(U) \) obeys \( \text{RC}(5/\sigma_r^2(X), \frac{25}{4} \kappa^2 \| X \|_2^2, 1/10 \sigma_r(X)) \). The convergence result in Equation (3.4) now follows from Lemma 5.6. All that remains is to prove Lemmas 5.7 and 5.8.

5.2.1 Proof of the regularity condition for the average function \( F \) (Lemma 5.7)

We first state some properties of the Procrustes problem and its optimal solution. Let \( U, X \in \mathbb{R}^{n \times r} \) and define \( H := U - XR \), where \( R \) is the orthogonal matrix which minimizes \( \| U - XR \|_F \). Let \( A \Sigma B^T \) be the SVD of \( X^T U \); we know that the optimal \( R \) is equal to \( R = AB^T \). Thus,

\[
U^T XR = B \Sigma B^T = (XR)^T U,
\]

which shows that \( U^T XR \) is a symmetric PSD matrix. Furthermore, note that since

\[
H^T XR = U^T XR - R^T X^T XR = (XR)^T U - R^T X^T XR = (XR)^T (U - XR) = (XR)^T H,
\]

(5.10)
we can conclude that $H^T X R$ is symmetric. To avoid carrying $R$ in our equations we perform the change of variable $X \leftarrow X R$. That is, without loss of generality we assume $R = I$ and that $U^T X \succeq 0$ and $H^T X = X^T H$.

Note that for any $U$ obeying $\text{dist}(U, X) \leq \frac{1}{4} \|X\|$ we have
\[
\left\| (UU^T - XX^T) U \right\|_F \leq \left\| UU^T - XX^T \right\|_F \|U\| \leq \frac{5}{4} \|X\| \left\| UU^T - XX^T \right\|_F.
\]
Using the latter along with the simplifications discussed above, to prove the lemma (i.e. (5.5)) it suffices to prove
\[
\langle (UU^T - XX^T) U, U - X \rangle - \frac{1}{20} \left( \left\| UU^T - XX^T \right\|_F^2 + \left\| (U - X) U^T \right\|_F^2 \right) \geq \frac{\sigma_r^2(X)}{4} \|U - X\|_F^2 + \frac{11}{5 \kappa^2} \left\| UU^T - XX^T \right\|_F^2. \tag{5.10}
\]
Equation (5.10) can equivalently be written in the form
\[
0 \leq \text{Tr}((H^T H)^2 + 3H^T H H^T X + (H^T X)^2 + H^T H X^T X
- \left( \frac{1}{20} + \frac{1}{5 \kappa^2} \right) [ (H^T H)^2 + 4H^T H H^T X + 2(H^T X)^2 + 2H^T H X^T X ]
- \frac{1}{20} [ (H^T H)^2 + 2H^T H H^T X + H^T H X^T X ]
- \frac{\sigma_r^2(X)}{4} H^T H).
\]
Rearranging terms, we arrive at
\[
0 \leq \text{Tr}(c_1(H^T H)^2 + c_2 H^T H H^T X + c_3(H^T X)^2 + c_4 H^T H X^T X - \frac{\sigma_r^2(X)}{4} H^T H)
= \text{Tr}(\frac{c_2}{2\sqrt{c_3}} H^T H + \sqrt{c_3} H^T X)^2 + (c_1 - \frac{c_2}{4c_3})(H^T H)^2 + c_4 H^T H X^T X - \frac{\sigma_r^2(X)}{4} H^T H).
\tag{5.11}
\]
Here the constants $c_1, c_2, c_3, c_4$ are defined as
\[
c_1 = \frac{9}{10} - \frac{1}{5 \kappa^2}, \quad c_2 = \frac{27}{10} - \frac{4}{5 \kappa^2}, \quad c_3 = \frac{9}{10} - \frac{2}{5 \kappa^2}, \quad c_4 = \frac{17}{20} - \frac{2}{5 \kappa^2}.
\]
Since $\kappa^{-1} \leq 1$, it is easy to verify that (a) $c_i > 0$ for $i = 1, \ldots, 4$, and (b) $c_1 \leq \frac{c_2}{4c_3}$. To prove (5.11) it thus suffices to prove,
\[
\|H\|^2 \leq \frac{c_4 - \frac{1}{4}}{c_2/4c_3 - c_1} \sigma_r^2(X).
\]
Using the fact that $\frac{c_4 - \frac{1}{4}}{c_2/4c_3 - c_1} \geq \frac{1}{16}$ for $\kappa \geq 1$ we see that the proof is complete by assuming $\|H\| \leq \frac{1}{4} \sigma_r(X)$.  

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5.2.2 Proof of gradient concentration (Lemma 5.8)

Define $\Delta := UU^T - XX^T$. Then,

$$
|\langle \nabla f(U) - \nabla F(U), V \rangle| = \left| \sum_{k=1}^{m} \langle A_k, \Delta \rangle \langle A_k, VU^T \rangle - \mathbb{E}[\langle A_k, \Delta \rangle \langle A_k, VU^T \rangle] \right|
$$

\[= \langle a \rangle \left| \sum_{k=1}^{m} \langle A_k, \Delta \rangle \langle A_k, VU^T \rangle - \langle \Delta, VU^T \rangle \right| \]
\[\leq \delta_{4r} \left\| UU^T - XX^T \right\|_F \left\| VU^T \right\|_F. \]

where (a) follows from our assumption that $A$ is isotropic and (b) follows from Lemma 5.1, since $\text{rank}(\Delta) \leq 2r$ and $\text{rank}(VU^T) \leq r$. This proves the first part of the lemma. To prove the second part, by the variational form of the Frobenius norm, we have

$$
\| \nabla f(U) - \nabla F(U) \|_F = \sup_{V \in \mathbb{R}^{n \times r}, \|V\|_F \leq 1} \langle \nabla f(U) - \nabla F(U), V \rangle
$$
\[\leq \delta_{4r} \left\| UU^T - XX^T \right\|_F \sup_{V \in \mathbb{R}^{n \times r}, \|V\|_F \leq 1} \left\| VU^T \right\|_F.

The result now follows from $\left\| VU^T \right\|_F \leq \|V\|_F \|U\| \leq \|U\|.

5.3 Proof of initialization (Equation (3.3))

Using Lemma 5.1, we can conclude that $\rho(A)$ from Lemma 5.2 is bounded by $\rho(A) \leq 2\delta_{4r} \leq 1/5$. Setting $Z_0 = 0_{n \times n}$ and applying Lemma 5.2 to our initialization iterates, we have that

$$
\left\| Z_\tau - XX^T \right\|_F \leq (1/5)^\tau \left\| XX^T \right\|_F \leq (1/5)^\tau \|X\| \|X\|_F.
$$

From Lemma 5.4, we have that

$$
\text{dist}(U_0, X) \leq \frac{\sqrt{2}}{\sigma_r(X)} \left\| Z_\tau - XX^T \right\|_F \leq \sqrt{2}(1/5)^\tau \|X\|_F.
$$

Hence, if we want the RHS to be upper bounded by $\frac{1}{4} \sigma_r(X)$, we require

$$
\sqrt{2}(1/5)^\tau \|X\|_F \leq \frac{1}{4} \sigma_r(X) \implies (1/5)^\tau \leq \frac{\sigma_r(X)}{4\sqrt{2} \|X\|_F}.
$$

Since $\|X\|_F \leq \sqrt{\tau} \|X\|$, it is enough to require that

$$
\tau \geq \log(\sqrt{\tau} \kappa^2) + 2. \quad (5.12)
$$

Similarly, it is easy to check that if $\tau$ satisfies (5.12), then

$$
\left\| Z_\tau - XX^T \right\| \leq \|Z_\tau - XX^T\|_F \leq \sigma_r^2(X)/4,
$$

also holds.
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References


A Proof of Lemma 5.4

Define \( H = U - XR \). Similar to the discussion at the beginning of Section 5.2.1, without loss of
generality we can assume that (a) \( R = I \), (b) \( U^T X \succeq 0 \). and (c) \( H^T X = X^T H \). With these
simplifications, establishing the lemma is equivalent to showing that

\[
\text{Tr}((H^T H)^2 + 4H^T H H^T X + 2(H^T X)^2 + 2X^T X H^T H - \eta H^T H) \geq 0
\]  

(A.1)
holds with \( \eta = \frac{1}{2(\sqrt{2} - 1)\sigma_r(X)} \). We note that
\[
\text{Tr}((H^T H + \sqrt{2}H^T X)^2 + (4 - 2\sqrt{2})H^T H H^T X + 2X^T X H^T H - \eta H^T H) \\
= \text{Tr}((H^T H)^2 + 4H^T H H^T X + 2H^T X)^2 + 2X^T X H^T H - \eta H^T H).
\]
Hence, a sufficient condition for (A.1) to hold is
\[
(4 - 2\sqrt{2})H^T X + 2X^T X - \eta I_r \succeq 0. \tag{A.2}
\]
Recalling that \( H^T X = U^T X - X^T X \), and that \( U^T X \succeq 0 \), we have
\[
(4 - 2\sqrt{2})H^T X + 2X^T X - \eta I_r = (4 - 2\sqrt{2})U^T X + (2 - (4 - 2\sqrt{2}))X^T X - \eta I_r \\
= (4 - 2\sqrt{2})U^T X + 2(\sqrt{2} - 1)X^T X - \eta I_r.
\]
Since \( U^T X \succeq 0 \), to show (A.2) it suffices to show
\[
2(\sqrt{2} - 1)X^T X - \eta I_r \succeq 0 \iff X^T X \geq \frac{\eta}{2(\sqrt{2} - 1)} I_r.
\]
The RHS trivially holds, concluding the proof.

**B Proof of Lemma 3.4**

From RIP and the assumption that \( \delta_{2r} \leq 1/10 \), we have
\[
\|Z_\tau - XX^T\| \leq \|Z_\tau - XX^T\|_F \leq \sqrt{\frac{10}{9}} e_\tau.
\]
By Weyl’s inequalities, this means that
\[
\sigma_r^2(X) \geq \sigma_r(Z_\tau) - \sqrt{\frac{10}{9}} e_\tau. \tag{B.1}
\]
Lemma 5.4 ensures that
\[
\text{dist}(U_0, X) \leq \sqrt{\frac{3}{2}} \frac{1}{\sigma_r(X)} \|Z_\tau - XX^T\|_F.
\]
We can upper bound the RHS by the following chain of inequalities,
\[
\sqrt{\frac{3}{2}} \frac{1}{\sigma_r(X)} \|Z_\tau - XX^T\|_F \overset{(a)}{\leq} \sqrt{\frac{3}{2}} \frac{1}{\sigma_r(X)} \sqrt{\frac{10}{9}} e_\tau \\
\overset{(b)}{\leq} \sqrt{\frac{3}{2}} \frac{1}{\sigma_r(X)} \frac{1}{2\sqrt{6}} \left( \sigma_r(Z) - \sqrt{\frac{10}{9}} e_\tau \right) \\
\overset{(c)}{\leq} \sqrt{\frac{3}{2}} \frac{1}{\sigma_r(X)} \frac{1}{2\sqrt{6}} \sigma_r^2(X) = \frac{1}{4} \sigma_r^2(X)
\]
where (a) follows from RIP, (b) follows since \( e_\tau \leq \frac{3}{20} \sigma_r(Z) \) implies that
\[
\sqrt{\frac{10}{9}} e_\tau \leq \frac{1}{2\sqrt{6}} \left( \sigma_r(Z) - \sqrt{\frac{10}{9}} e_\tau \right),
\]
and (c) follows by (B.1).