N. Wahl, "Compactifying String Topology."

Plan: "Physics"
- Geometric
- Algebraic

\rightarrow \text{(essentially)} \text{ some compactifications of moduli space parameterizing operators.}

I. "String sections" a la Knudsen-Penner

\begin{itemize}
  \item \text{String sections should be parameterized by the moduli space of Riemann structures on } M_{g,p+2}.
\end{itemize}

\text{Combinatorial model: for } M_{g,p} \text{ [Harer, Penner].}

\text{(open) cell decomposition indexed by "filling" arc families. }
\text{meaning complement = union of polygons.}

\text{Point to such a cell = weighted arc family}
\text{going to boundary of cell } \implies \text{ weight } \rightarrow 0 \implies \text{arc disappears.}
\text{Non-compact: not all arcs are allowed to disappear, b/c of "filling" condition.}

\text{Compact variant: [Kauf}man\text{]: work w/ arcs from in to out boundaries, with total weight } = 1 \text{ at each incoming boundary, b/c no filling condition.}
Fact: (Ehres~berg's ~twin) ~Kaluzn's are familiar give a cell decomposition of Bors~hime~r's ' harmonic compactification ' of $\overline{\mathbb{H}^2}$ This compactification allows degeneration of 1st order curve in map in boundary has to come out out

3) String topology a la Chas~Sullivan (joint work in progress with Hingston)

Chas~Sullivan: $C^\ast(LM) @ C^\ast(LM) \rightarrow C^\ast(LM)$

General string operation: $C^\ast(LM)^p \rightarrow C^\ast(LM)^{p \times 2}$ parameterized by

- collection of points on p loops
- partitions of the points: an equiv. relation $\sim_p$ of $S^2$.

If $e \in \Delta^1 \times \ldots \times \Delta^1$, parameterizes points on simplex beyond basepoint.

E.g. $C^\ast(LM)^{p \times 2}$:

- "Read~off loop": $\frac{1}{2} S^2 \rightarrow H^8 \overset{\sim}{\rightarrow} D$
Get: \( LM_\mathcal{P} \times \Delta^k_1 \times \cdots \times \Delta^k_\mathcal{P} \leftarrow \sum \left\{ (x, t) \mid y(x, t) = y_D(t) \right\} \]

cell in the space

\( = \text{loops satisfying the intersection pattern determined by } D. \]

Get: a space parameterized by pictures like \((\ast)\) loop diagrams

Algebraic model: [Jones, W. Westerland]

related to: Kontsevich-Schommer-Costello

[Radler-Zeinalian]

Needs: Assume \( \pi, M = 0 \), work over field (essentially \( \mathbb{Q} \)).

Jones: \( \text{H}^*(LM) \cong \text{HH}^*(C_\mathcal{P}(M), C^*(M)) \)

Know: If \( M \) is a manifold,

\( H^*(M) \) satisfies P.D. and "Freudenthal algebra"

so \( C^*(M) \) is some stupid version of this.

8 the structure on \( C^*(M) \) should give structure on \( \text{HH}^* \).

Thin [WW]:

Type of algebra

\[ \text{MAGCHNE} \]

\[ \text{chrs cplx of natural operators on } C_\mathcal{P}(A, j_A) \text{ alg. of that type} \]

satisfies some unusual property.

e.g., commutative.

Freudenthal (Gelfand, \( \text{IBL} \), ?)

[Define by prop, rep, assoc., \( \cdots \) ]
$H^J(LM) \rightarrow C^J(M)$

Example: For Frobenius algebras,

this complex of operations includes Kauffman's arc systems, e.g.,

Sullivian diagrams, $\otimes$ composes to the same homology,

e.g., $H^\otimes_\ast(M^\otimes)$. 

[KLANT]: For commutative Frobenius algebras, this extends

the cellular complex of loop diagrams, (but don't know if homology

isomorphic).

Fact: [LANRECHTS-STANLY]: Over $\mathbb{Q}$, $C^\otimes(M) \cong$ CDA

as strict comm. Frobenius (e.g., finite-dimensional) algebra, but not necessarily Frobenius algebra

nor $C^\otimes(LM) = C_\ast(C(M), C^\otimes(M)) \cong C_\ast(A, A)$

Get this way string operators parametrized by $C_\ast(M^\otimes)$ and loop diagrams.

Moduli space $M, M^\otimes$, and loop diagrams

\[ \text{NEW HOMOLOGY FROM COMPACTIFICATION, } \text{Rank: } A \text{ is unique } \text{if } M \text{ is framed.} \]

\[ \text{in } H^\ast \text{ kills all the stable homology, } 0 < s < \frac{2\pi}{3} \]

\[ \text{Sure bad news: there were way too many of these operations we knew!} \]

A preserves the BV structure, acting non-trivially on $H^\ast(LS)$

(\$ \cong \text{family of stable classes;} \$

Support examples: $A \otimes \emptyset$.

Auravägen 17, 182 60 Djursholm, Stockholm
Examples: the Gromov–Hausdorff coproduct.

\[ C^*_{\infty}(LM) \rightarrow C^*_{\infty}(LM) \otimes C^*_{\infty}(LM) \]

Cutting loops at all times \( t \).

(Originally defined mod contract loops, but admits a lift.)

Algebraically:

\[ C^*_{\infty}(A, A) \otimes C^*_{\infty}(A, A) \rightarrow C^*_{\infty}(A, A) \quad \text{can be written} \]

\[ (a_0 \ast \tau a_u) \otimes (b_0 \ast \tau b_v) \rightarrow a_0 \tau b_0 \omega a_u \ast \tau b_v \ast a_0 \tau b_0 \]

Generically:

\[ LM \times \Delta \rightarrow \{(x, t) \mid x(t) = x(0)\} \]

Locally:

\[ LM \times S^1 \rightarrow LM \]

Views \( LM \times S^1 \)

Right:

Not a usual braiding, but close enough, we still define a Thom diagram up.

E basis part:

This is the \( \left( a \ast b \right)_G H = \left( a \ast b \right)_G H \)

\[ = \left( * a \ast b \right)_G H \nu_G H \varepsilon \]

\[ + \left( * a \ast b \right)_G H \]

(Only works for \( G = \mathbb{R} \))

Need to extend left \( \ast \) for \( \mathbb{R}^+ \).