Vesna, $\partial \tilde{E}_2$ part

$\mathbb{D} : \text{Top}_X \to \text{Top}_P$

$P_n X \subset D_n X = \Omega \infty D_n X = \Omega \infty (\mathcal{C}_n \times X^n)_{k \Sigma_n}$

Under nice conditions, $X \to \text{holim}_n P_n X$

Gives:

$\Rightarrow \text{GSS (Goodwillie spectral sequence)}$

$$E_2 \text{ term } p = \pi_1^* D_n X \Rightarrow \pi_1^* X$$

$E_2(X)$ \hspace{1cm} stable \hspace{1cm} unstable \hspace{1cm} is often more complicated than stable.

How can we compute

$E_1(X), E_1(S^k)$?

$H_*(\mathcal{C}_n, H_*(D_n S^k \mathcal{F}_p)) = H_* (\mathcal{C}_n) \otimes H_* (S^k)$

\[ \text{L S.S.} \]

$H_* ((\mathcal{C}_n \times S^k)_{k \Sigma_n})$

[Adams-Habour]

\[ \text{computed as a module over} \]

$\Delta^* S\Delta \text{ of skew polynomial algebra} \text{ itself, call by to use Adams s.s. to compute } \pi_\ast$
Chromatic approach: (p-local)

"decomposing into frequencies"

\( p, V_1, V_2, \ldots \)

 Moral: Type \( m \) complexes know about \( V_m \)-periodicity.

\( \pi_n S^k \) into \( V_m \)-periodic parts.

Thm: \( D_n(S^k), k \text{ odd} \)

\[
\begin{align*}
\Rightarrow & \quad n \neq p^i \\
\Rightarrow & \quad D_p(i(S^k)) \text{ has type } b \quad \text{\& knows about } v_i \text{-periodic homotopy in } S^k. \\
\text{\quad (technically not finite complexes, but usual statement)}
\end{align*}
\]

(Will)

Says for spheres, Taylor tower gives better decomposition of homotopy groups.

(use our sphere decomposes as to odd spheres, so morally the same statement is true with this observation).

Look at the following sequence of limits:

\[
\begin{array}{c}
\text{Id} \xrightarrow{} \Omega E \xrightarrow{H} \Omega \Sigma_1 S^Q \\
\Sigma : X \mapsto X \times X \\
\sim : \Sigma \Omega \Sigma_1 X \xrightarrow{} \Sigma V^1 X^1 \\
\text{\& is adjoint to } \tilde{\Pi}.
\end{array}
\]

(Snaith splitting)
Now, \( p = 2 \).

2-locally: fiber sequences

\[
\cdots \rightarrow \Omega \Sigma \Omega P_{n} \left( S^{k} \right) \rightarrow P_{n} \left( \Sigma \Omega \Sigma \Omega \left( S^{k} \right) \right) \rightarrow \Omega \left( \Sigma \Omega \Sigma \Omega \left( S^{k} \right) \right) \rightarrow \cdots
\]

and

\[
\cdots \rightarrow \Omega D_{n} \left( S^{k+1} \right) \rightarrow D_{n} \left( \Sigma \Omega \Omega \left( S^{k} \right) \right) \rightarrow D_{n} \left( \Sigma \Omega \Omega \left( S^{k} \right) \right)^{2} \rightarrow \cdots
\]

Lemma: \( F : \text{Top} \rightarrow \text{Top} \) is a (finite) homotopy functor,

(Stably injective for all \( i \)?)

Then

\[
P_{n} \left( F S^{k} \right) \sim P_{n} \left( \Omega \Sigma \left( F \right) \left( S^{k} \right) \right)
\]

\[
D_{n} \left( F S^{k} \right) \sim \begin{cases} D_{n/2} \left( \left( F \right) \left( S^{k} \right) \right) & \text{if } n \text{ even} \\ * & \text{if } n \text{ odd} \end{cases}
\]

If: Chain rule.

Want to show that \( D_{n} \left( S^{k} \right) \sim \ast \) if \( n \neq 2^{j} \) for some \( j \).

If \( n \) odd, by fiber sequence (*)

\[
\cdots \rightarrow \Omega \Sigma \Omega D_{n} \left( S^{k+1} \right) \rightarrow \Omega^{2} \Sigma \Omega D_{n} \left( S^{k+2} \right) \rightarrow \cdots
\]

\[
\cdots \rightarrow \Omega^{\infty} \Sigma \Omega D_{n} \left( S^{k} \right) \sim D_{n} \left( \Omega^{n} \left( S^{k} \right) \right)
\]

\( \ast \) is not necessary b/c \( S^{k} \) takes finite to finite.
by induction hypothesis: \\

for some \( n = s \cdot 2^j \) s odd, \( j \) & all \( k \) spheres \\

show true for \\
\( s \cdot 2^{j+1} = 2n \).

\[
D_{2n} S^k \rightarrow \Omega D_{2n} (S^{k+1}) \rightarrow \Omega D_n (S^{2k+1})
\]

\[12\]

So, can iterate again, apply same \[ \vdots \]

argument to get \\

\[
D_{2n} S^k = D_n (\Omega D_{2n})(S^{k+1}) \simeq *
\]

Recall: \\
\[ Z_n = (\sum_{p} S^p X_n)^{vee} \text{ susp. } \text{ st. } (x) \]

- \( X_n \) is \( \text{unbased suspension partition complex} \)
- \( n = [3, \ldots, n] \)
- \( X_n \) is poset of non-trivial partitions \( (> 1 \text{ and } \emptyset \text{ in set is a partition}) \)

and \( X_n \) is \( |K_n| \).

And recall \( Z_n \) is \( \simeq (V S^{n-1})^V \), \( K_n \) is \( V S^{n-3} \). \\
From earlier talk.
When \( n \geq 2 \), \( K_n = \emptyset \). (No non-trivial partitions)

\[ \implies Q_2 = S^{-1} \text{ w/ trivial } \Sigma_2 \text{-action.} \]

Goal: we'll find a smaller complex \( B_k \) such that \( K_{p^k} \sim B_k \), (equivalently \( \delta \)-locally).

And, the way, show \( (K_n) \sim \ast \), \( n \neq p^k \).

\[ \sim \sqrt[n]{S^{n-1}} \text{ (exponentially smaller).} \]

(B\(k\) will turn out to be the Tits building for \( GL_k(\mathbb{Z}/p) \), something studied in representation theory.)

\( B_k \): simple set of flags in \( (\mathbb{F}_p)^k \), e.g., \( s\)-simplices are:

\[ 0 \subset V_1 \subset V_2 \subset \cdots \subset V_s \subset F_p^k \]

subspaces \& inclusions

face/edge, maps are forgetting or doubling.

(Bruhat-Tits building for this \( k \)-s.)

How to see \( B_k \) as part of \( K_{p^k} \)?

Thee: \( \mathbb{F}_p^k \sim \mathbb{R}^k \) \& think of flags as giving you a partition.

(chief subspace, complement of rest, etc.)
Now, let $n = \text{general}$

$K \xrightarrow{\text{bijective \ \ under \ preserving \ map}} \mathfrak{S} \in \text{set of stabilizers}$

$\uparrow$

$x \rightarrow H_x \subset \Sigma_n$

$\text{up to any $g \in G$},$

$H_x \cong \Sigma_{n_1} \times \ldots \times \Sigma_{n_k}$

So,

$K_n \cong [1^k].$

Thus, we're reduced to studying $\text{Rep } \Sigma_n.$

More generally:

$\mathfrak{C} \triangleq \text{collection of subgroups } \Sigma \leq G \text{ (closed under conjugation)},$

(not necessarily closed under subgroups),

and

$X \in G \quad \xrightarrow{\text{isotropy groups}} \quad \text{Is}_0(X) \triangleq \{H \text{ which stabilize simplices of } X\}.$

We say $X$ has $\mathfrak{C}$-isotropy if $\text{Is}_0(X) \subseteq \mathfrak{C}.$

We say $X \rightarrow Y$ is $\mathfrak{C}$-equivariant if $\Sigma$ subgroups $H$ of $\mathfrak{C},$ $X^H \rightarrow Y^H.$
Prop: There is a unique \( C \)-approximation

\[ \exists \: X_C \rightarrow X \]

s.t. it is a \( C \)-equivalence & \( X_C \) has \( C \)-isotopy.

\( C \) = all subgps, \( X_C = I \)
\( C = \{ G \} \), \( X_C = XG \)
\( C = \{ \{ e \} \} \) \( X_C = EG \times X \)

Define \( EC = \{ (\ast) \} \).

\( E \) approx. to a point, i.e.

\( C \) is a poset, so can take \( EC \rightarrow |C| \).

This is not an equivalent equivalence, but is a way.

\[ (EC)^H \rightarrow \begin{cases} \ast & H \in \mathcal{C} \smallsetminus \{ \} \\ \{ H \in C \mid H \subset H \} & H \text{ has indel point} \\ \{ \} & \text{or terminal, if?} \end{cases} \]

\( \mathcal{F} = \text{all of non-trivial, non-trivial subgps of } \mathcal{E}_n \)

Clearly \( \emptyset \in \mathcal{F} \) if it turns out that \( E\mathcal{F} \overset{\sim}{\rightarrow} \mathcal{E} \mathcal{F} \)

This means that \( \mathcal{K}_n \approx |\mathcal{F}| \).
The last step is the most complex step (Chap. 4 rep. theory).

Let

$$E': \text{ maximal elementary subgps } \sim (\mathbb{Z}/p)^i \quad \subseteq \quad E_n.$$  

(An important non-trivial fact: if we only care about trivial stuff, we only need to look at it.)

And $E' = \mathbb{E}^n$.

Claim: $EE' \rightarrow EF$ is an $\mathbb{E}^n$-approximation.

Proof: Exercise 1 (fill details).

Borel-equiv. mod $\beta$ cohomology:

$$H^*([E_n \times \cdot \cdot \cdot \times E_n], F_\beta).$$

Final diagram:

$$\begin{align*}
E' & \xrightarrow{\sim} E' \\
E^\prime & \xrightarrow{\approx} E \quad \xrightarrow{\approx} \quad C \\
E^\prime & \xrightarrow{\approx} E \quad \xrightarrow{\approx} \quad \{x, \beta \text{ iso. equiv.} \}
\end{align*}$$

So it’s a $p$-adic equivalence after taking homotopy groups.
Now, suppose \( n \neq p^j \).

Counting \( \Rightarrow \exists' = \exists \) (by \( \exists \) is not transitive on \( F \))

\( \Rightarrow C = x \).

Otherwise: the difference between \( \exists, \exists' \) is exactly the building \( B_j \).