H. Tanaka, Factorization Homology and Manifold Calculus

\[ F: \text{Open}(M)^{op} \to \mathcal{C} \quad (\text{Fix } n = \text{dimension of our manifolds}) \]

instead, \[ F: \text{Mfld}/M^{op} \to \mathcal{C} \]

Def: \text{Mfld} is a Top-enriched category

\( \text{Obj: n-mflds} \) (some technical assumptions, e.g. no non-gauged surfaces)

\( \text{Hom}(X, Y) = \text{Emb}(X, Y) \) \text{space of embeddings} \]

A map in \( \text{Mfld}/M \) is

\[ X \to Y \quad \text{(non-shift)} \]

\[ Y \prec X \quad \text{need a path in space of embeddings to } M \]

Now, study

\[ F: \text{Mfld}^{op} \to \mathcal{C} \quad (\text{"manifold calculus"}) \]

Factorization homology \quad \text{Manifold Calculus}

\[ \mathcal{A}: \text{Mfld} \to \mathcal{C} \quad F: \text{Mfld}^{op} \to \mathcal{C} \quad \text{some category over spaces} \]

Def: Disk \subset \text{Mfld} is the full subcat. whose object are of the form \( \frac{1}{i} \mathbb{R}^n \) for \( 0 \leq i < \infty \).
Defn: $A : Mfd_0 \rightarrow C$

$\uparrow$

$\text{Disk}_j$

$\text{Disk} \leq j$

$\text{Disk} \leq 0$

$(\text{homotopy})$

the left Kan extension of $A / \text{Disk} \leq j$

along the inclusion functor $\text{Disk} \leq j \hookrightarrow Mfd_0$

will be denoted $T_j A$, and is the $j^{th}$ polynomial approx to $A$.

By universal property of left Kan extensions, have

$\operatorname{holim} \left( T_0 A \rightarrow T_1 A \rightarrow T_2 A \rightarrow \cdots \right) = T_\infty A$

We say $T_\infty A(M)$ is the factorization homology of $M$ with coefficients in $A$, written as $\int_M A$.

Ex: $Mfd_0 \rightarrow \text{Spaces}$

$M \mapsto M$ (forget manifold str.)

$\left( f : M \rightarrow N \right) \mapsto \left( f^* M \rightarrow \int_N \right)$ is analytic.

Pf: This is represented by $\operatorname{Hom}(R^n, -)$.

$M_{\text{flat}} \mapsto \text{respects framing}$.
$$\mathcal{F}_0 A (\mathcal{M}) = \text{Emb}^b (\mathcal{M}, \mathcal{M}) \otimes \text{Emb}^b (\mathbb{R}^n, \mathcal{M}) \quad \text{(left hand extension is)}$$

$$\subset \text{Emb}^b (\mathbb{R}^n, \mathcal{M}) \subset \mathcal{M}$$

In fact this is $T_1$ too, i.e. $A$ is linear.

Similarly, the fiber hom $(\bigcup_j \mathbb{R}^n, \mathcal{M})$ is poly. of degree $2^j$.

**Ex.** Let $U = \mathbb{R}^n \setminus \{0\}$. Then $\text{Emb}^b (U, \mathcal{M})$ is not analytic.

**Pr.** $\text{Emb}^b (U, U) \neq \text{Emb}^b (\mathbb{R}^n, U)$,

(possibly that $T_{\mathcal{M}} \text{Emb}^b (U, \mathcal{M}) = \text{Emb}^b (\mathbb{R}^n, \mathcal{M})$.

(? maybe incorrect?)

Let's restrict our attention to functors $(\text{Spaces}, \otimes)$

$$A: (\mathbb{Mfld}, U) \rightarrow (C_\infty, \otimes) \quad (\mathcal{C}_\infty, \otimes)$$

which are sym. monoidal.

**Obs:** $A|_{\text{Disk}}$ defines an $E_n$-algebra

$$A (\mathbb{R}^n) \otimes A (\mathbb{R}^n) \rightarrow A (\mathbb{R}^n)$$
A $DBh_{fr}$ for defines an $A_{\infty}$ $(E_1)$ algebra.

 muito: the inclusion of \( \phi \) into \( \mathbb{R}^n \) gives initial structure.

Then (exercise): Let \( A \) be sym. toroidal. (Francis, Ayala-Francis - T.)

Given \( M = N_0 \cup N_1 \cap V x \mathbb{R} \).

\[
\bigcup_{\pi} A \cong \bigcup_{N_0} A \otimes \bigcup_{N_1} A
\]

\( \mapsto \mathbb{D}(S^1) \).

module structure & shifting \( \delta \).

Ex: \( M = S^1 \) \( (n = 1) \).

\[
S^1 = \bigcup_{U} \mathbb{C}
\]

\[
S_{S^1} A = \bigcup_{R} A \otimes \bigcup_{R} A
\]

\[
\bigotimes_{R \cup \mathbb{R}} A \otimes (\mathcal{O}_R)^{op}
\]

This is Hochschild homology.
**Defn:** A homology theory for $n$-mfdls is a functor

$$H: \text{Mfd} \rightarrow (\mathcal{C}, \otimes)$$

s.t.
1. Continuous functor (map on hom spaces is a map of Top).
2. $H \rightarrow \otimes$.
3. $H$ satisfies excision.

**Thm (Furus):**

$$\text{En-alg}(\mathcal{C}) \rightarrow \{ \text{Homology theories for } n\text{-mfdls} \}$$

**MAd calculus:**

$$F: \text{Mfd}^0 \rightarrow \mathcal{C}$$

\[ \Downarrow \]

$$\text{DBK}^0$$

\[ \Downarrow \]

$$\text{Disk}^0$$

\[ \Downarrow \]

The/Defn:

$$T: F \mid \text{Disk} \rightarrow \text{Right Kan extension of } F \mid \text{Disk}$$

Along $\text{Disk} \rightarrow \text{Mfd}$.

The in poly. approx. to $F$. 
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\[ \text{Tor}_{\text{Disk}}(\text{Emb}(\cdot, M), A) \]

\[ \text{Res}_{\text{Disk-mod}}(\text{Emb}(\cdot, M), F) = \text{Tor}_0(F)(M) \]

(From Cattel.)

\[ \text{Then: } \text{Tor}_0 F(M) \cong \text{Hom}_{\text{R-mod-Disk}} \left( \text{Emb}(\cdot, M), F \right) \]

\[ \text{Assume } F \text{ symmetric.} \]

\[ \text{Claim Then: Given } M = N_0 \cup N_1, \text{ Then } F(M) \cong \text{cobord}(\text{full}(N_0), \text{full}(V \times R), \text{Tor}_0 F(N_1)) \]

\[ \text{Defn: } \text{A homology "theory" for } \text{n-mflds is a functor} \]

\[ H : \text{Mfd}^{op} \rightarrow (\mathcal{C}, \otimes) \]

\[ (1) \text{ cls. full, } (2) \mathcal{C} \text{-2 tensor, } (3) \text{ satisfies (6) axioms} \]

\[ \text{Claim Then: Consider } \]

\[ E_n \circ \text{adj}(\mathcal{C}) \rightarrow \{ \text{Homology theories for } \text{n-mflds} \} \]

\[ E_n \circ \text{-adj}(\mathcal{C}) \rightarrow \{ \text{"derived mflds"} \} \]

\[ \text{ex: } \text{Easy homology} \]

\[ \text{Tor}_{\text{ACS}}: \text{AVSA} \rightarrow \text{THA}(A) = \text{Tor}_{\text{ACS}} A \]