Day 5 Talk 2: James,

Fukaya Cat. of Torus Fibration s

\[ p: X \to Y \]

\[ X \text{ symplectic, } Y \text{ compact base} \]

\[ p \text{ locally trivial, Lagrangian torus fibration} \]

Kontsevich-Soibelman 2001:

Relate \( F(X) \) to \( \mathcal{O}_Y \)-modules,

\[ \mathcal{O}_Y \text{ sheaf of (}\Delta\text{) algebras on } Y \]

\[ \text{Novikov ring} \]

\[ D_\bullet(F(X)) = D^b(\text{coh } \mathcal{O}_Y \text{-mod}) \]

Affine structure on \( Y \)

Def: Any affine structure on \( Y \) consists of an

atlas \[ \{ \psi_i : U_i \to \mathbb{R}^n \} \]

s.1. \[ \psi_i \circ \psi_j^{-1} \in \text{GL}(n, \mathbb{Z}) \times \mathbb{R}^n \]

\( Y \text{ integral affine } \to p: X \to Y \)

\[ X = T^* Y / T^*_Z Y \]

\[ \text{local system of integral cotangent vectors.} \]
Now, given $X \rightarrow Y \rightarrow Y \rightarrow \mathbb{H}_1(fiber, \mathbb{Z})$ local system on $Y$, locally:

- Take a basis of sections, $\xi_1, \ldots, \xi_n$.
- Represent them by $(n+1)$ closed submanifolds $\Sigma_1, \ldots, \Sigma_n \subset X$.

(Pwsh: these are all Lag'n torus fibrations which admit a Lag'n section).

Basepoint $x_0$. 

**Words:**

$$
\begin{aligned}
\omega & \mapsto \gamma \mapsto \int_{x_0}^{x} \omega = \int_{x_0}^{x} \rho^{-1}(y) \wedge \Sigma_i \\

\Lambda := \sum_{i=0}^{\infty} \sum_{c_i T^{a_i}} |c_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lim_{i \to \infty} \lambda_i = 0 \rangle

\text{Define } val(\Sigma_c; T^{a_i}) := \min \{ \gamma | x \}

\text{To construct } \Theta_Y, \text{ construct } \Theta_{R^n}, \text{ along with an action}
\end{aligned}
$$

$$
\begin{aligned}
\Theta_{R^n}(U) = \left\{ \sum_{k_1}^{\infty} \ldots \sum_{k_n}^{\infty} a_{k_1} \ldots z_{k_1} \ldots z_{k_n} \mid a_{k} \in \Lambda, \kappa = (k_1, \ldots, k_n) \right\} \\
\forall y \in U, \lim_{N \to \infty} \inf_{1 \leq |k| \leq N} \text{val}(a_k) + \langle k, y \rangle = \infty.
\end{aligned}
$$
How do you think about this?

$f \in \Omega^\infty_{\mathbb{R}^n}(U)$ formally a Laurent series over $(\Lambda^\infty)^n$

Last condition says $f(z_1, \ldots, z_n)$ converges in $\Lambda$

whenever $\{\text{val}(z_i)\}_{i=1}^n \in \mathbb{U}$.

Paul: comment when non-Archimedean analogic

growth order larger than symmetry.

\[ F_k : \Omega^\infty_{\mathbb{R}}(\mathbb{R}) = \sum_{\lambda \in \Lambda} a_\lambda z^\lambda \mid a_\lambda \in \Lambda \]

as $k \to \pm \infty$, the order

(valuation) of $a_\lambda$ should grow

more than linearly

the analogue of an entire holomorphic fn.

(want convergence in Novikov ring --)

What is the action?

$A \in \text{GL}(n, \mathbb{Z}) ; \quad y_1 \mapsto Ay$

$\sum_{k \text{ multi-index}} a_k z^k$ converges in $U$ $\iff$ $\sum_{k \text{ multi-index}} a_k A^k z^k$ converges in $AU$.
\( b \in \mathbb{R}^n, y \mapsto y + b \)

\[
\sum a_k z^k \iff \sum a_k T^{-k} \bar{z}^k
\]

conv. in \( U \)

needs to see that semi-direct product acts on them?

Exercise

Now we've defined \( O_\gamma \).

\( \text{col. } O_\gamma \text{-mod} \)

(Pauli simplest case, take the circle, we've defined the analytic space associated to the Tate family of elliptic curves -- sheaves correspond to something else)

\( \text{over Spec } \mathbb{C} \rightarrow \)

want \( F(X) \rightarrow O_\gamma \text{-mod} \)

Consider Lagrangian submanifolds of \( X \), s.t.

\( p|_L : L \to Y \) is an unramified covering.

think of a section or multisection.

call resulting category \( \text{Fun}_{\text{ram}}(X) \).

\[
L \in \text{Fun}_{\text{ram}}(X) \quad \text{[First, suppose } p|_L : L \to Y \text{ is 1:1]}.
\]

\( F(L) \) - locally free \( O_\gamma \)-module.

Over a sufficiently fine open covering \( U \) of \( Y \),
\[ F(L)(U_i) = \Omega_Y(U_i) \]

choice of isomorphism corresponds to choice of \( f \in C^\infty(U_i) \) s.t. \( L \cap p^{-1}(U_i) = \text{graph of } df \mod T^*_Z Y \).

**Ex:**

![Diagram](image)

Change \( f \mapsto f + l \), where \( dl \) is a section of \( T^*_Z Y \) multiply by \( \exp(l) \)

i.e. \( l = c + \langle m, y \rangle \)

\( := T^c T T^*_Z y_i \)

integral

\( Y^* \)

If \( p: L \to Y \) is not 1:1, take direct sum locally of sheaves associated to each sheet.

**Ex:**

![Diagram](image)
Ex: $Y = \mathbb{R}^n / \mathbb{Z}^n$, $X = \mathbb{C}^n / \mathbb{Z}^{2n}$, $g_1 = 0$, abelian variety.

$L_i = \text{section given by } df$, $f = \text{quadratic form on } \mathbb{R}^n$

$s.t. df \text{ takes integral values on } \mathbb{Z}^n$.

$L_0 = \text{0-section } \text{ of } F^* (L_0, L_i), \text{ concentrated in degree } \text{index of quadratic form } f$.

Would like to see that

$rk \text{ } H^{1*} (L_0, L_i) = rk \text{ } \text{Ext}^1 (F(L_0), F(L_i))$

Concretely:

$F(L_0) = \mathcal{O}_Y$, $b/c$

$L_0 = \text{graph of } f$ for $f \in \mathcal{C}^\infty (Y)$ (e.g. $f = 0$)

$H^0 (L_0)$

$HF^{0*} (L_0, L_i) = \Lambda$

$HF^1 (L_0, L_i) = 0$

$\text{Ext}^0 (F(L_0), F(L_i)) = \Lambda$

$H_0 (F(L_i))$.
Suppose $\sum a_k z^k$ is some section.

In passing from neighborhood of 0 to a nbhd of 1, shift: $y \rightarrow y + 1$.

\[
\frac{1}{z} y^2 \rightarrow \frac{1}{z} (y-1)^2 = \frac{1}{z} y^2 - y + \frac{1}{2}
\]

\[
\sum a_k z^k \rightarrow \sum a_k T^{-k} z^k \rightarrow \sum a_k T^{-k-1} T^{1/2}
\]

(multiplying) by exp (difference)

In order to define global section, need

\[
a_{k-1} = a_k T^{-k} T^{1/2} \rightarrow \text{gives recurrence}
\]

determining all coefficients from a single one, say $a_0$.

So $h^0(F(L_i)) \leq 1$.

\[
\text{val}(a_{k-1}) = \text{val}(a_k) - k + \frac{1}{2}
\]

\[
\text{val}(a_k) \sim k^2
\]

\[
\Rightarrow \text{convergence}
\]

get a global section.

$h^0(F(L_i)) = 1$
In this example

\[ \begin{array}{c}
\text{Here}, \quad \text{get}
\end{array} \]

\[
\text{recurrence } a_k \sim an+2, \text{ so }
\]

\[
2 \text{ degs. of freedom.}
\]

\[
\text{Note: } \exp(-y + \frac{1}{2}) \approx \mathbb{Z}^{-1} T^{1/2}.
\]

\[
\mathcal{F}_{\text{Fun, ram}}(X) \to \mathcal{O}_Y\text{-mod embedding},
\]

\[
\text{When } X \text{ tors, } \mathcal{F}_{\text{Fun, ram}} \text{ will split-generate.}
\]

\[
\text{(full subcat of)}
\]

\[
D\mathcal{F}_{\text{Fun, ram}}(X) \sim \text{dg enhancement complexes of } \mathcal{O}_Y\text{-modules.}
\]

\[
X \text{ algebraic, supposed to get everything.}
\]

\[
\text{There's some kind of mirror torus fibration picture underlying it.}
\]