Day 4 Talk 4: Discussion

Possible topics:
1. Twisted complexes and what does "derived" mean?
2. Signs (of madness)
3. Grading on HF^*
4. Open-closed string theory
5. Obstructions & curvature / has to do w/
6. Koszul duality and motives
7. Immersed Lagrangians

Curved A_oo-structures:

\[ 0 \rightarrow A \]
\[ \text{hom}_A(X,Y) \]
\[ m^d : \text{hom}_A(k_{-d}, X) \otimes \text{hom}_A(k_0, X) \rightarrow \text{hom}_A(k_0, X) \]

Branches:
\[ \text{hom}_A(X_0, X_1)[2-1] \quad (d \geq 0) \]

\[ H^1_A(m_A^0) = 0 \text{ in } H^3_A(X, X) \]
\[ m_A^1(m_A^1(x)) = m_A^2(y_0, x) \oplus m_A^2(x, y_0) \]

Ex: \[ E \rightarrow M \text{ vector bundle w/ connection} \]
\[ A = \Omega^*(M, \text{End}(E)) \]
\[ m' = d_A, \quad m^2 = \lambda, \quad m^0 = F_0 \]
\[ \mathfrak{X} = \mathcal{S}_{\bullet}^0(H, \text{End}(E)) \]
\[ y^1 = \bar{\partial} \gamma, \quad y^0 = F^0 \gamma. \]
\[ d^2 = 0 \text{ has interesting solutions (thanks to structure of nilpotent products) } \]
\[ d^2 = 1 \text{ less interesting solutions (get a splitting)} \]
\[ y^0 = 0 \Leftrightarrow \text{vector field vanishes at } 0 \]
(thinking about these as cauchy)\]

But maybe the v.f. vanishes somewhere else!

Try to change/translate infinitesimally to extract something meaningful.

Technical condition: \( \mathfrak{X} \) defined over \( \mathbb{Z} \) \( \mathbb{F} \), \( y^0 \) of order \( t \), \( y^1 \) of order \( t \) (use formal deformation theory to make it go away).

\[ \text{Ob} (\tilde{\mathfrak{X}}) = \{ (x, \alpha) | x \in \text{Ob } \mathfrak{X}, \alpha \in \text{Hom}^2(\mathfrak{X}, x) \}, \]
\[ y^0 + y^1(\alpha) + y^2(\alpha, \alpha) + \cdots = 0 \]

Inhomogeneous Maurer-Cartan
Alternatively, just consider category of all such objects, & throw away obstructed guys. (For call this "filtered" Ax so alg.)

You can do this & think complexes at once, but not related.

Version of this you can do which hasn’t really been written down.

If connection is appropriately flat, induced connection on End bundle is flat.

\[ \rho \in \mathfrak{g}^2(\mathcal{A}, \mathcal{A}) \]
\[ d\rho = 0. \text{ of order } t \]

\[ \text{assume } \mathcal{A}_{[p]} \text{ (not convex)} \]

\[ \text{Ob } (\mathcal{A}_{[p]}) = \{(x, \alpha) \mid -y^0 + y^1(\alpha) + y^2(\alpha, x) + \cdots \} \]

Why is this important?

\[ (M, w) \in \mathfrak{g}(H) = 0 \]

A small, b/c no hol. discs of area \( \Omega \)

(\& power controlled by \( \omega(\beta) \)).

\[ \text{\& boundaries of } \]

\[ \text{holo. discs} \]
Try to solve
\[ y^0 + y^1 (x) \] gradually, step by step.

For example:
\[ L^3 \subset H_6, c_1 (H) = 0 \]
\[ H^1 (L) = H^2 (L) = 0 \]
\[ \Rightarrow \text{unique object in } \mathcal{F} (\ldots ) \]
(i.e. can solve, but solve uniquely)

For other cases, generally an internal 4d space of paths/choices

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**Physics**: Objects \( (L, A \in \Omega^1 (L, R)) \),
\[ w_1 |_L = 0, \ F_A = 0 \] (to guarantee good conditions in twisted \( \sigma \)-model).

This is before instanton corrections.

Actually, what’s true is \( \frac{i}{2\pi} F_A = - \mathrm{PD}(y^0) \).

Max’s examples for \( \frac{i}{2\pi} \) current

\( \rho = \text{const.} \), so can transport away.
\( (c_1 (H) \neq 0, \text{so not graded any more} ) \)
Immersed Lag's (Joyce, 

\[ L \hookrightarrow M \quad \text{(Lagrangian immersions)} \]

\[ \mathcal{CF}^*(L, L) = \mathcal{C}^*(L) \oplus \bigoplus_{(x, y) \in \pi} \mathbb{Z} \langle x, y \rangle \]

\[ \text{More complex: } \psi(x) = \psi(y) \]

\[ \text{(why ordered pairs?)} \]

\[ \text{when you flow, 2 intersection points.} \]

\[ \eta^0 = \text{holo discs} \]

\[ \text{Claim: everything else is the same.} \]

\[ u : \mathbb{H} \xrightarrow{\text{hol}} M \]

\[ \text{lim}_{s \to 0^+} \tau(s) = x \]

\[ \text{lim}_{s \to +\infty} \tau(s) = y \]

\[ u(0) = u(1) \]

\[ \text{ex: seems unobstructed to naked eye, but look at differential.} \]
Morally, think of all solutions to Maurer-Cartan or arising from isomorphies?

**Signs and gradings**

\[ L_0, L_1 \subset \mathcal{M} \]

\[ \mathcal{P} = \{ \gamma : [0, 1] \to \mathcal{M} \mid u(0) \in L_0, u(1) \in L_1 \} \]

\[ \mathcal{P} \to U_{\infty} / O_{\infty} \]

1. \[ H^1(U_{\infty} / O_{\infty}) = \mathbb{Z} \rightarrow H^1(\mathcal{P}) \text{ obstruction to grading} \]
2. \[ H^2(U_{\infty} / O_{\infty}, \mathbb{Z}_2) = \mathbb{Z}_2 \rightarrow H^2(\mathcal{P}; \mathbb{Z}_2) \text{ obstruction to signs} \]

1. \[ 2c_{rel}^1 \in H^2(M \times [0, 1], L_0 \times \mathbb{R}_3 \cup L_1 \times \mathbb{R}_3) \]

\[ \downarrow \text{transgression} \quad \uparrow \text{try to kill this} \]

\[ H^1(\mathcal{P}) \]

2. \[ w_2(L_0) \otimes w_2(L_1) \in H^2(L_0) \otimes H^2(L_1) \]

\[ \downarrow \text{evaluation} \]

\[ H^2(\mathcal{P}; \mathbb{Z}/2) \]

(Some version of family Atiyah-Singer index theorem.)
(1) Some sort of Calabi-Yau condition, but more.

Killing $2\zeta_1$:

$$(\Lambda^n \mathcal{C} T \mathcal{M})^\otimes 2 = \zeta_1$$

("you don't want to know it's dead, you want to just kill it explicitly")

Let's say we're killing $c_1$; choose complex $n$-form

$\eta \in (\Lambda^n \mathcal{C} T \mathcal{M})$.

Let $c \in TM$:

$$\eta_c : \mathcal{M} \rightarrow \eta_c \in H^1(L)$$

$$\eta_c = [\alpha_c : L \rightarrow S^1]$$, $\alpha_c(x) = \eta_c(TL_x)$.

Choose a real-valued phase, a lift

$\tilde{\alpha} : L \rightarrow \mathbb{R}$ of $\alpha$.

A surface + trivial class of $TM$ = oriented foliation. $\tilde{\alpha}$ gives you a way of rotating $TL$ into the foliation.
Thomas-Yau: apply this to uniqueness of special Lagrangian.

This reproduces the classical grading in $\mathbb{F}_x$ Morse theory.

\underline{Signs:} Pick a spin structure (not an actual class, an actual spin structure) and spin automorphism acts on $HF$ by $(-1)$ — very messy.

\underline{Families index theory}

\[ P \to Y/0 \]

\[ \Sigma \varphi \to \Sigma(Y/0) \cong \mathbb{Z} \times B_{SO} = Fredholm profile \]
compute using Atiyah-Singer

Next obstruction:

\[ H^2(P; \mathbb{Z}/2) \rightarrow H^1(\mathbb{Z}/2; \mathbb{Z}/2) \rightarrow H^0(\mathbb{Z} \times \mathbb{Z}/2, \mathbb{Z}/2) \]

det. line of Frobenius operator

\[ \psi_1 \text{ (det line bundle) } \]

If curves, do you have a 1-param. family of det. lines, what is the det. line bundle?

trivial or not? family index thing.

(Atiyah-Singer p1. 5)
Open-Closed String Theory

How does $F(-)$ fit into an O-C string theory?

Formal theory:

Open-closed: framed little disc opersad.

$O(1) = \begin{cases} \circ \circ \end{cases} \cong \mathbb{R}$

$O(2) = \begin{cases} \circ \circ^{\circ} \end{cases} \cong \mathbb{T}^2$

$O(3) = \begin{cases} \circ \circ \circ \end{cases}$

Second by shearing disc/Gutting etc., aligning marked points.

Algebra over $H_\ast (\mathbb{C})$: BV-algebra

$\nabla$ graded vector space, differential

$\Delta: \mathbb{V} \rightarrow \mathbb{V}[-1]$

$\mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V}$ commutative, associative.
7^3 classes give x y, x Δy, Δx y, Δ(x y); (∆^2 = 0).

The failure of ∆ to be a derivative is a Lie bracket,

\[ \{ x, y \} = \Delta(x y) - x \cdot \Delta y - \Delta x y \]

Jacobi from ∆(∆) = 0.

This is a genus 0, closed string topological field theory (i.e., a BV algebra).

\[ \text{open-closed: } V_\text{od} \rightarrow V \rightarrow W_{\text{od}} \]

\[ V, W \text{ carries a graded symmetric pairing} \]

\[ \mathbb{C} \rightarrow W_{\text{od}} \]
compactification

open-closed top. str. theory.
bro-colored, \( (\mathbb{Z}, \text{ cyclic}) \) operad

parmke boundary pts.

: \( C \rightarrow \text{Wolg} \)
gives \( W \) the structure of an \( A_{\infty} \) algebra which is cyclic \( (\text{-Y}) \)

(Fuk categories also have this, up to homotopy, after lots of work)

homotopy BV-algebra

can look at chains of these guys, but get no new info. (some sort of functorial result)
mixed guys: what happens to these?
gives a map of homology BV algebras (similar to Sheel's talk).

\[ \nabla : \text{CC}^*(W, W) \to H^3(W, W) \]
\[ (\theta \text{ has structure of BV algebra, relgular) = \text{some work} \]
\[ \text{counit } \delta = \text{Coom's operator} \]
\[ \text{Sheel's product} \]

should be a formal consequence of product structure of these moduli spaces.

(idea: \( C^*(M) \approx \text{CC}^*(L/L) \))

\[ \sqrt{\text{odd}} \to \text{W odd} \]

Can recover some closed string shift from W, i.e. set \( \nabla = \text{CC}^*(W, W) \) \( (\delta) = \text{id} \).

(don't require \( (\delta) \) to be a quasi-iso, i.e. \( W = 0 \))

but there are cases in which this is the case (open-closed TFT := alg over chains on bicircular algebra.)
What actually happens (imprecise)

Moduli spaces are further compactified to Deligne-Mumford spaces

Closed

open

nothing happens

degen

Closed - strong Operad $\mathcal{O}$.

$\mathcal{O}(1) = \bigvee \cup \cong pt$

$\Delta$ deg

$\mathcal{O}(2) = \bigcup \cong pt$ (consider all degeneracies)

$\mathcal{O}(3) = \overline{M}_{0,4} \cong S^2$ (Paul's claim)
genuinely new 3-fold product

open-string cycli $A_{oo}$-structure. $V$

closed-string sector: $\text{Conf FT}$ (cohomological field theory)

(oo of generators).

Kontsevich: in general, some relations between points, not enough.

Dubrovin, Givental

(open-closed)

string sector

$V \rightarrow W$

still have $V \rightarrow C^*(W, W)$

 Coh FT homotopy BV

If this is a quasi-isom. $\Rightarrow$ Connes boundary operator vanishes

$\Rightarrow$ spectral sequence from $H_{t} t^*(W, W)[u]$

to $H_{C} C^*(W, W)$ degenerates (Connes 6).

is first differential.
We would like the spectral sequence to degenerate anyway i.e. get a map of_htpy Coh$^*$'s. 

Kontsevich conjecture: if $A$ is scheme "smooth," get this degeneration.

within four kähler geometry; degeneration of Hodge-deRham forms.

On $F$, we have all these sheaves, in particular.

$\Delta = 0 \Rightarrow \Sigma^* = 0$ or $V$.

i.e. $V \rightarrow \mathbb{C}^*(W, W)$ map of dgla's bracket $= 0$.

$\Rightarrow$ family of Am-$*$-algebras over $V$ (big Fukaya category).

(Really useful for, e.g., toric varieties).

(i.e. get a sheaf of Am category over $V$, fiber at $0$ is usual Fukaya category).
Can, e.g., deform by \( c_1 \) which has some geometric significance.

As a standard deformation,

deg k piece \( \rightarrow \) rescale by \( j_k \).

suppresses higher order terms.

On the whole, unsatisfactory.

Can't really reconstruct \( V' \) from \( (C^\infty (U, \mathbb{C})) \).

In principle, know what you have to do to upgrade hotopy \( BV \rightarrow \text{CohFT} \).

Additional piece of info required (to do \( U \) killing \( \Delta \)). First manifestation is a power series in the variables \( w \) values in \( HC_* (W, \mathbb{C}) \).

Given \( W \), have different choices of data, give different CohFT's. Group that acts on those is \( \infty \text{-dim} \), like a loop group of a symplectic or \"Eulerian\'s twisted loop group,\" cf. Kevin Costello.
Costello knows how to add in the extra data.

There is absolutely a higher genus version of this.