Day 3 Talk 1: Calabi-Yau Quantum Cohomology

I. J-holomorphic spheres $(M,\omega)$ 2n-dim \'l.

$\pi_2(M) \to H_2(M)$ 1st exc.

image are called "spherical classes"

Thm: $\exists$ a subset $J_{\text{reg}}(A) \subset J_2(M,\omega)$

of 2nd category st. $\forall J \in J_{\text{reg}}$

$M^*(A;J) = \{ u : S^2 \to M \mid u \text{ J-holom.} \}$

simple \[ [u] = A \]

is a smooth unfold of dim $= 2n + 2c_1(A)$

First note: $\text{PSL}_2(\mathbb{C}) \to M^*(A;J)$, &

this \ is non-compact, so our moduli space is non-compact.

Def: $M^*(A;J) \times \mathbb{C}P^k \overset{ev}{\to} M^*(A;J)$

$\downarrow \text{ev}$

$M \times \mathbb{C}P^k \to \text{dim}$

$2n + 2c_1(A) + 2k - 6$

Def: $(M,\omega)$ is semi-positive if $\forall A \in H_2(M)_{\text{sp}}$

if $c_1(A) > 0$, $c_1(A) > 3n \Rightarrow c_1(A) > 0$.

Thm: $(M,\omega)$ semi-pos. then $\exists J_{\text{reg}}(M,\omega) \subset J_2(M,\omega)$

of 2nd category st. $\forall A \in H_2 \text{ with } c_1(A) > 0, J \in J_{\text{reg}}(M,\omega)$
Then \( \text{ev} : M^*_{0,k} (A; J) \to M^* \)
gives a pseudo-cycle of dimension \( 2n + 2c_1 + 2k - 6 \).

\text{I.e.,} \quad f : V^d \to M \quad \text{such that} \quad \dim (\text{ev}(V) - f(V)) \leq d - 2.

\underline{II. 2 point \ GW \ invt.}

\( \dim 3 \). \underline{Notation:} \( H^k (X) = \text{free part of } H^k (X; \mathbb{Z}) \)
\( H^k (X) = \text{Hom} (H_k (X; \mathbb{Z}) , \mathbb{Z}) \)
\( a \in H^i (M), b \in H^j (M), c \in H^k (M), H \in H^{spk} (M) \)
\( GW^3_A (a, b, c) \)
\( = \text{ev} \circ (\alpha \times \beta - \gamma) \quad (\alpha = PD (a) \cdots ) \).

\( \dim (M^*_D_{1,3}) = 2n + 2c_1 (A) \)

Need \( \deg a + \deg b + \deg c = 2n + 2c_1 (A) \)

\underline{Rule:} \( GW^3_A \) is graded commutative in \( a, b, c \)
(exercise on orientability).

\( \underline{Ex.} \ A = 0 \text{ mod } m, \text{ so } \text{ev} = [\Delta M] \in H_{2m} (M^3) \)
\( GW^3_A (a_1, a_2, a_3) = [\Delta M], a = \int_M u_1 v_2 u_3 \)
Ex. \((p^m, \omega_a)\), \(H_2(Cp^n) = \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow [Cp^1]\)

\(H^\infty : g(L) = n+1\)

\[GW_m^3 (a, b, c) \text{ doesn't vanish only when}\]

\[
\sum \deg = 2n + 2m(n+1), \quad \text{i.e.}\]

\[m = 0, \sum \deg = 2n\]

\[m = 1, \sum \deg = 4n+2\]

Remains to compute

\[GW_m^3 (p^i, p^j, p^k) = 1 \iff i + j + k = 2n+1\]

\(p = PD[L]\)

\[GW_m (p, p^n, p^n) = 1.\]

Conclusion:

\[
GW_m^3 (p^i, p^j, p^k) = \begin{cases} 1 \\ 0 \end{cases} \begin{array}{c} m = 0, i + j + k = n \\ m = 1, i + j + k = 2n+1 \end{array}
\]

(point is that \(i + j + k = 2n + 2m(n+1) \geq 6n+1\))

III. Quantum Cohomology

Assume \(M\) is monodrome \((c_2(A) = 2c_1(A), \ g > 0)\)

As abelian groups, \(\oplus H^*(M) = H^*(M) \otimes \mathbb{Z} [g, g^{-1}]\)

\[
N = \min |c_1(A)| \quad \text{deg} \; q = 2N.
\]

\[
A, c_1(A) \neq 0
\]
Ring structure:

\[ a \in H^4(M), \ b \in H^2(M), \]
\[ a \ast b = \sum_{A \in H_2} (a \ast b)_A \varepsilon \]
\[ H_{k+l-2c_1(A)} \]

Define \((a \ast b)_A\) to satisfy
\[ \langle (a \ast b)_A, c \rangle = GW^3_A(a, b, c) \]
\[ \langle x, y \rangle := \int_M x \cup y \quad 2n + 2c_1(A) = \Sigma \text{deg} \]
\[ \Rightarrow 0 \leq c(A) \leq 2n \]
\[ \Rightarrow \text{sum is finite} \]

Extend similarly to \(QH^* : QH^* \otimes QH^* \to QH^*\)

Ranks:
1) this is distributive
2) " - graded commutative.
3) Thus: is associative.

Back to \(P^m:\)
\[ p_i \ast p_j = \sum_{mL} (p_i \ast p_j)_{mL} \varepsilon \]
\[ c_1(mL)/(m+1) \]
\[ \langle (p_i \ast p_j)_m p_k \rangle = GW^3_{mL}(p_i, p_j, p_k) \]
\[
p^i \ast p^j = \begin{cases} \prod_{i+j \leq n} \beta_i & \text{if } i+j \leq n, \\ \emptyset & \text{if } n \leq i+j \leq 2n \end{cases}
\]

Not quite right.

So, \(QH^*(P^n) = \mathbb{Z} \left[ q, q^\frac{1}{2} \right] / (p^{n+1} = 2)\)

Sketch of associativity:

\[
QH^* \circ QH^* \circ QH^* \rightarrow QH^*
\]

\[
\triangleq abc \rightarrow (a \ast b) \ast c.
\]

\[
(a \ast b) \ast c = (b \ast c) \ast a = a \ast (b \ast c)
\]

\[
\Delta (a \ast b) \ast c \quad \text{is graded commutative.}
\]

To do so, now:

\[
\langle (a \ast b) \ast c \rangle_A, d \rangle = \langle \langle \sum_B (a \ast b)_B \ast c \rangle_A, d \rangle
\]

\[
= \langle \sum_B ((a \ast b)_B \ast c)_{A-B}, d \rangle
\]

\[
= \sum_B GW_{A-B}^3 ((a \ast b)_B, c, d).
\]
\[ = \sum_{B} GW_{\beta, A}^{2, 2} \left( \alpha_1, b, c, d \right) - B \]

\[ = GW_{\alpha, \beta}^{(0, 1, 0, 2)} \left( \alpha, b, c, d \right) \]

This guy is graded commutative, so we're done.

Coeff: (M, w) any closed sym., define
\[ A_{\text{Novikov ring of w}} \]
formal sums \[ \tau = \sum A \left( A \right) e^{A} \]
\[ \forall h \in H_{2} \]
\[ \exists A \in H_{2}(M) \mid \tau(A) \neq 0 \quad \exists \omega_{(A)} \leq C \quad \forall C \in \mathbb{R} \]
\[ a \star b = \sum_{A} (a \star b)_{A} e^{A} \]

If (M, w) is CY, i.e. \( c_{1} = 0 \) on spherical classes, can use the universal Novikov ring.
\[ \Lambda^0 = \{ \lambda \in \mathbb{R}^3 : \exists \epsilon \in \mathbb{R} \mid 3 \epsilon \geq 0 \} \]

\[ a \ast b = \sum_{\mathcal{A}} (a \ast b)_\mathcal{A} + c(A) \]

for general case:

\[ \Lambda = \Lambda^0 \cup [q, q^{-1}] \]

\[ a \ast b = \sum_{\mathcal{A}} (a \ast b)_\mathcal{A} + c(A) + q \cdot \frac{1}{q} \]