Fix $(M, \omega)$. For $L_0, L_1 \subset M$, we have

$$
HF^*(L_0, L_1) = H^*(CF^*(L_0, L_1), \delta = \eta^{'})
$$

Def'n: Assume $L_0 \not\cong L_1$. Then,

$$
CF^*(L_0, L_1) = \bigoplus \bigwedge^* \times \times \times
$$

$$
\Lambda = \sum_{k=0}^{\infty} a_k \tau^k \mid a_k \in \mathbb{Z}/2, \gamma_k \in \mathbb{R}, \gamma_k \to \infty
$$

(Doing this with coverings is painful; need choice of basept below if those get in your way).

$$
\partial(x) = \sum_y n(x, y) y ; n(x, y) = \sum_{y \in \mathbb{R}} (\eta^y(x, y))
$$

$$
n_y = \# \text{ of isolated } \gamma_y \leftarrow \leftarrow \gamma_x \text{, with }
$$

$$
E(u) = \int_{L_0} u^* \omega = \gamma.
$$

Main property: invariant under exact Lagrangian (Hamiltonian) isotopies of either $L_0$ or $L_1$. 

(NB: Non-exact bodies change area of disks, so maybe prevent things from canceling or vice versa).

\[ L = L_0 = L_1 \]

\[ H^* (L, L) = H^*(S'; \Lambda). \]

\[ \alpha > 0 \]

\[ H^*_\alpha (L, L) = 0. \]

---

Some notes: Need to perturb \( L_i \) to be transverse, can always do this w/ liem. isotopies. Also, differential always has \( \xi > 0 \), b/c

\[ F(v) = \int_{R^c[0,1]} u^* w = \int_{R^c[0,1]} w(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}) ds dt \]

\[ = \int_{R^c[0,1]} |\frac{\partial v}{\partial t}|^2 ds dt > 0 \]

So can you restrict to \( \Delta' = \xi \xi^{>0} \) powers? Yes, but not an isotopy invariant.
(as the maps giving quasi-isomorphisms from Hamiltonian isotopies have $t^{<0}$ terms).

Product structures.

Take $L_0, L_1, L_2$ in general position.

$\text{CF}^*(L_1, L_2) \otimes \text{CF}^*(L_0, L_1) \xrightarrow{m^2} \text{CF}^*(L_0, L_2)$

$(x, y) \mapsto \sum_{z} n(x, y, z) z$

$n(x, y, z)$ counts maps (taking into account energies)

Lemma: $m^2$ is a chain map.

Proof-by-picture

\[ = 0 \]
\[ \mu' \circ \mu'^2 + \mu'^2 (1 \otimes \mu' + \mu' \otimes 1) = 0. \]

Limit process above has:

\[ \text{bubble} \]

\[ \text{principal component.} \]

Define the Donaldson–Fukaya category

- objects: \( L = M \)
- morphisms: \( \text{Hom}(L_0, L_1) = \text{HF}^*(L_0, L_1) \)
  (if necessary perturbing to make them transverse),
- composition induced by \( \mu^2 \).

Fact: This category is unital (even though you're perturbing, \( I \) is identity morphism on homology level).

Fact: \( \text{Symp} \) acts on this category

Hom. isotopy ("weak action")

NB: If you up this \( J \) to a 2-group, acts nicely.
Hamilton's are well-defined up to canonical isom.

Introduced by Donaldson following ideas of Segal, but can't do anything with it.

- Gives a disconnected picture of structure ... can't really say much...

Pass to the chain level

Simplified partial version:

Fix a finite ordered collection \((L_1, \ldots, L_m)\) of Lagrangian submanifolds. Define the
Directed Fukaya category \(F \to (L_1, \ldots, L_m)\) as follows:

- **Objects**: \(L_1, \ldots, L_m, \mathbb{F} \subseteq \text{CF}^*(L_i, L_j), \ i \leq j\)
- **Morphisms**: \(\text{Hom}(L_i, L_j) = \bigcup \text{ke}_L, \ i = j\)
  \(i = j\) part is formal, this avoids dealing with \(\text{Hom}(L_i, L_i)\) need to throw away \(\text{Hom}(L_j, L_i)\) if \(j > i\) to ensure we can't compose back to \(\text{Hom}(L_i, L_i)\).

This has the structure of an Aoo category.
$u^1 : \text{hom}(L_i, L_j) \to \text{Hom differential}(L_i, L_j)$

or zero

$u^2 : \text{hom}(L_i, L_h) \otimes \text{hom}(L_i, L_j) \to \text{hom}(L_i, L_h)$

the holomorphic triangle product (corresponding extensions which make $e_{L_i}$ into a unit)

$u^d : \text{hom}(L_{i_{0n}}, L_{i_d}) \otimes \cdots \otimes \text{hom}(L_{i_{2n}}, L_{i_1}) \to \text{hom}(L_{i_{0n}}, L_{i_d})$

non zero only if $i_{0n} < \cdots < i_{d}$

Definition of $u^d$ : Given $u^d \in L_{i_{0n}} \cap L_{i_d}$,

$x_2 \in L_{i_0} \cap L_{i_2}$, \ldots (inputs) and $x_0 \in L_{i_{0n}} \cap L_{i_d}$,

and $\nu > 0$, consider :

$R^{d+1}(x_0, \ldots, x_d)^\nu = \xi(S, z_0, \ldots, z_d, u)$

5 Rem. surface isomorphic to a closed disc $z_0, \ldots, z_d \in \partial S$ distinct cyclically ordered boundary pts

$u : S \setminus \{z_0, \ldots, z_d\} \to M$ $J$-hol. maps with the following boundary and asymptotic conditions :
This comes with a forgetful map

$$\mathbb{R}^{d+1}(x_0, ..., x_d) \rightarrow \mathbb{R}^{d+1}$$

(over killing)

$$n(x_0, ..., x_d)^\gamma := H\mathbb{R}^{d+1}(x_0, ..., x_d) \in \mathbb{Z}/2$$

(Count isolated points, throw rest away)

(Over killing)

$$n(x_0, ..., x_d) \equiv \sum_{\Lambda} n(x_0, ..., x_d)^\gamma \in \Lambda$$

$$m_d(x_0, ..., x_d) \equiv \sum_{x_0} n(x_0, ..., x_d)$$
Thus: $F \rightarrow (L_1, \ldots, L_n)$ is an A∞ category.

Proof by picture: Uses compactification

$$\overline{R^{d+1}(x_0, \ldots, x_1)} \rightarrow R^{d+1}$$

Generically, in $\mathbb{R}^n$, with faces of not corners.

$$0 = \partial(\mathcal{Q}) = \sum \mbox{pts.}$$

$d+1$ boundary points
$e+f = d+1$
$e+f \geq 2$,

$$+ \sum \mathcal{Q}$$

$i$-th marked pt.

Note: Don't need to treat these two separately, the differential is just $\partial$, part of a sequence (contrast to looking at A∞ operad, where need to put in $\partial$ by hand, rest of $u_i$ are operad structure).
Getting an actual $A_\infty$ category refining the
Donaldson–Fukaya category. This problem already
(perturbations don't help!) occurs in Morse
theory!

Involves choices of perturbations.

For any $(L_0, L_1)$, choose $(L'_0, L'_1)$ which are
Ham. isotopic & transverse.

\[ \text{ham}(L_0, L_1) = CF^*(L'_0, L'_1). \]
Problem:

\[ \eta^2 : CF^*(L_1^-, L_2^+) \otimes CF^*(L_3^-, L_1^+) \to CF^*(L_0^-, L_1^+) \]

\[ \Lambda : \mathbb{R}[\bar{z}_0, \bar{z}_1, \bar{z}_2] \]
\[ \text{Lag}(m) \]

\[ \Lambda(\bar{z}) \text{ varies by Ham. isotopy} \]

Consider maps $u : D^2 \setminus \{\bar{z}_0, \bar{z}_1, \bar{z}_2\} \to M,$

\[ u(z) \in \Lambda \quad z \in \partial D^2 \quad (\text{moving boundary condition}) \]

\[ u \text{ J-holomorphic} \]
(obv. non-canonical)

Extend this to all Riem. surfaces that occur in a way that's consistent with compactification.

Then: The resulting $A_{\infty}$ structure is independent of all choices up to quasi-isomorphism.

When does this work as described? $\omega = 0, L$ exact

* $[\omega] = 0$, M noncompact but nice at $\infty$.

* $[\omega] = \pi_{1}(A), (A > 0)$ L "meridinal" or "Baker--Sawon".

* $[\omega] = \pi_{1}(A), (A < 0)$, $2\pi_{1}$, divisible by $n-1$.

* $A = 0$, $n + \pi_{1}(A) - 3 \leq 2n = \dim_{C}(M)$ the $C_{1}$

* $c_{1} = 0$, $n = \dim_{C}(M) \leq 2$. L's with vanishing Hodge.

min. alg. str. (borderline) In each case, only a particular class of L's is allowed. codim.

call this: the last case.

Note: The reason why it fails is always the same!

Each of these cases has different short term fixes.

**no joke for generality, but call it a phenomenon.**

Have to choose a point outside the walls to specify object in $F(M)$.