Day 2 Talk 2: Hom

Setting: \((M, \omega)\) \(\omega\) symplectic form \(\omega = 0\) \(\text{symm-deg.}\)

\(L_0, L_1 \subset M\) Lagrangian submanifolds
\(\omega|_{L_0}, \omega|_{L_1} = 0\)

\(L_0 \cap L_1\)

Pictures:
1. \(M = \mathbb{R}^2\)
   \[\text{[Diagram of } \mathbb{R}^2\text{]}\]

2. \(\dim M = 4\)
   \[\text{[Diagram of 4-dimensional manifold]}\]

**Maslov Theory**

- The Manifold
  \[\Sigma M = \{ x \in C^\infty([0, 1], H) \text{ s.t. } x(0) \in L_0, x(1) \in L_1 \}\]

Choose \(p \in \Sigma M\), constant part \(\bar{p}\) \(\in L_0, L_1\),
\(\bar{x} \in \Sigma M\) can be written as \((x, [\bar{u}])\) where \(x \in \Sigma H,\)
\(\bar{x}^2 = \bar{x}, x\) a symmetry from \(\bar{p}\) to \(\bar{x}\).
The function
\[ u : [0, 1] \times [0, 1] \to M \]
\[ A(x, y) = \int_0^1 \int_0^1 u^* c \]

Exercise: \( A \) is well-defined.

The metric: Let \( J \) be an a.e. \( \mathbb{R} \) \( \alpha \cdot (-, J- \alpha) \) is a Riem. metric.
\[ T_x \mathcal{M} = \{ \text{vector fields along } x \text{ s.t. } X(x_0) \in \mathbb{R}, \quad X(x) \in L_1 \} \]
So, \( g(X, Y) = \int_0^1 \alpha(X(t), JY(t)) \, dt \).

Exercise: 3. The critical points of this \( A \) are given by constant points in \( \mathbb{R} \times L_1 \).

2. The gradient flow is given by maps
\[ u : [0, 1] \times \mathbb{R} \to M, \]
Jhol. maps such that
\[ u(1, t) \in L_1, \quad (\lim u(s, t) = \mathcal{E}) \]
\[ u(0, t) \in L_0, \quad (\lim u(s, t) = \mathcal{E}) \quad t \to -\infty \]
Given two almost C-manifolds $M_1, M_2$ with $\nu_1, \nu_2$, let $u : M_1 \to M_2$ be smooth if $du \circ \nu_1 = \nu_2$.

To define differential, we'd like to know $\dim M(p, q)$.

If $u$ is smooth from $p \to q$ then we can define the differential,

$$\delta p = \sum \nu_{p,q} \cdot \theta$$

such that $\dim(C(p,q)) = 0$.

Then $M(p,q) = M(p,2)/\text{parametrizations}$.

Paul:

Due to polarized manifolds, it's in general impossible to work w/ $\mathbb{C}$ cells, & when it's possible, requires some checking.

Today, we work w/ $\mathbb{R}/\mathbb{Z}$ cells.

Given some $u : M(p, q)$, how do we find the dimension of $M(1/2)$ on this component?
Answer: Maslov index

Small project: Learn about Maslov index.

\[ D^2 \xrightarrow{u} M \]

\[ u : (D^2, \partial D^2) \rightarrow (M, L) \]

We can pull back \( TM \) onto \( D^2 \)

\( TL \) onto \( \partial D^2 \).

Walking along the boundary gives us a map

\[ S^1 \rightarrow Gr^{\text{lag}} \]

Knowing \( \pi_1 (Gr^{\text{lag}}) = \mathbb{Z} \) we can compute the degree of this map.

How do we define this for \((M, L_0, L_1)\)?

R.M.T.
Translating $u^o TM$ gives $D^2 \times \mathbb{C}^n$.

Choose a picture like this.

Choose translation $(TL_0, TM)$ along both paths.

In $\mathbb{R}^2$ (Example)

Before reparametrization, this is dim $2$.

Maslov index $(u, p \rightarrow g) = 1$.

(dim. of $M(p, g)$ before reparam.)

What is dim $\mathcal{H}(p, x)$?

Ex. 2: $\frac{\text{dim}}{2} = 2$.

have some 1 degree of freedom
Take $M = T^* S$

$L_0 = \text{zero section}$

$L_1 = \text{graph}(df)$

Looking at a wine bottle, we see

$\tilde{M}(x,y) = \mathbb{Z}$ points

$\exists x = 2y = 0.$

$x \searrow \mathbb{Z}/2\mathbb{Z}$

$y \nearrow \mathbb{Z}/2\mathbb{Z}$

$HF^*(L_0, L_1) = \mathbb{Z}/2\mathbb{Z}$

Then \(^{(\text{floor})}\) let $M$ be a smooth manifold. For a "small fan," $HF^*(M, df)$ boo section in $T^* M$. \((\mathbb{Z}/2\mathbb{Z} \text{ orbs.}) \quad H^*(M; \mathbb{Z}/2\mathbb{Z}) \).
\[ f \text{ small near } E, \quad \forall x \in \mathbb{R} \setminus A \setminus \Sigma \text{ and } x \geq 0, \quad (r + 10f1, 100f) \leq 3. \]

Why?

\[ j \text{-hol. disc, } \text{ corresponds}\]

\[ \text{to TV line} \]

If \( \pi_2(M, \omega) = 0 \), \( M = \text{no section} \)

Corollary: \( H^\ast_1((L, L)) = H^\ast_2 L \).

Non-example:

\[ \varphi = 0, \quad \theta \equiv p, \quad e^{2t} \equiv 0. \]