Recall: $H^i(X, \mathbb{C}) \in R^i_{\text{hol}} \mathbb{C}$, $H^i(X, \mathbb{C})_S = H^i(X_S, \mathbb{C})$.

Analytic version:

$H^i(X, \mathbb{C})^{\text{an}} = H^i(X, \mathbb{C}) \otimes \mathbb{Q}_S$.

A sheaf locally looks like $\sum f_i \omega_i$ in a basis for cohomology hol. forms of a fiber.

$\mathcal{H}^2,2(S_X/S) := R^2\pi_* \mathcal{L}^p_{X/S}$

sections can locally look smooth relative $(0,2)$ forms and exact forms.

The Godement cover for viewing $H^i(X, \mathbb{C}) \subset H^i(X, \mathbb{C})^{\text{an}}$ as flat sections.

$G_{\text{H}} : H^i(X, \mathbb{C})^{\text{an}} \rightarrow H^i(X, \mathbb{C})^{\text{an}} \otimes \mathbb{Q}_S$.

In coordinates:

$\nabla_{G_{\text{H}}} (\sum_i [\omega_i]) = \sum_i d\omega_i \otimes [\omega_i]$. 
Example: \( E \) elliptic curve, e.g. \( \{ y^2 = x(x-1)(x-\alpha) \} \)

\[
\downarrow
\]

\[
S = \{ \alpha, \alpha \neq 0, 1 \}
\]

let \( \omega = \frac{dx}{y} \) (alg. differential on \( E \)).

let \( e_1, e_2 \) be basis for \( H^1(E, \mathbb{C}) \), \( \in \mathcal{S} \).

In local coords,

Can write \( \omega_2 = f_1(\alpha)e_1 + f_2(\alpha)e_2 \)

\( e_1, e_2 \) dual basis in \( H_2(E, \mathbb{Z}) \).

Then, \( f_i(\alpha) = \hat{e}_i(\omega_2) = \int_{\hat{e}_i} \omega_2 \in \text{classical period} \).

G-M conjecture measures

(1) How periods vary

(2) How span \( \langle \omega_2 \rangle = H^1 \) varies inside of \( H^1(E, \mathbb{C}) \) (rel. the canonical integral structure),

\( \mathfrak{D} : D_{GM}(F^p) \subseteq F^{p-1} \otimes \mathbb{Z}_p \). (Griffiths tranvsection)

\( F^p = \bigoplus_{r \leq p} H^r, \mathbb{C} \to H^i(X, \mathbb{C}) \)
§2. Period Domains:

A Hodge structure of weight $n$ in a real vector space $V_{\mathbb{R}}$

is the decomposition

$$V_{\mathbb{C}} := V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \quad \text{into}$$

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q} / H^{p,q} = H^{n-n}. \quad \text{Equivalently (in a way that works for families)}$$

A H.S. is a decreasing filtration

$$F^p \subseteq V_{\mathbb{C}} \quad \text{s.t.} \quad F^p \otimes F^{n-p+1} = V. \quad \text{To recover } H^{n-p} = F^p \cap F^{n-p}.$$

Attempt #1 at period domain:

"Space of admissible satisfying given conditions..."

- Lefschetz decomposition (relies on Kähler form) \( \Rightarrow \) "Primitive cohomology".

- On the primitive cohomology, there's a polarization (e.g.,

  - definite alternating or symmetric form, depending on degree)

A period domain is a moduli space of polarized Hodge structures with integral basis.

Ex. \( H \) is the moduli space of rank 2, weight 1 polarized Hodge structure with integral basis. \( = \text{Sp}(2)/\text{SO}(2). \)
Simply connected, so $H^i(\mathcal{E}, \mathbb{Z})$ is trivial.

Picking a basis, get a map $s \mapsto H^i \mathbb{Z}$ via

$$s \mapsto H^i(\mathcal{E}, \mathbb{Z})$$

gives lattice inside $\mathbb{C}$. If $S$ is not simply connected (or we don't choose basis), we get a map $S \rightarrow \mathbb{H} / SL_2(\mathbb{Z})$.

In general, $X$

$$s \mapsto H^i(X, \mathbb{R})_{\text{prim}}$$

Note: is a polarized Hodge structure, give a map

$$S \rightarrow \text{a period domain} / \text{arithmetic group,}$$

(target basis)

The target of the map is

$$K \backslash G(\mathbb{R})/G(\mathbb{Z})$$

where $G(\mathbb{R}) \cong GL_n(\mathbb{R})$ is the group sending

the polarized Hodge structure to polarized Hodge structures,

$K$ the stabilizer of your face. polarized Hodge structures,

$G(\mathbb{Z})$ is Aut($H^i(X, \mathbb{Z})$, $\mathbb{C}$ polarizations).
Rule: In many cases (e.g., if Griffiths transcendence is not vacuous
condition),
the period map cannot be surjective, i.e., it is injective
but is tangent to the Jacobian defined by Griffiths' transcendence.

Example:

\[
\begin{array}{c}
\text{any curve, } X \\
\text{Then (Pavlic): The } \Pi^1 \\
\text{period map}
\end{array}
\]
\[
\begin{array}{c}
\text{(curve } \longrightarrow \text{ Jacobian) determines the} \\
\text{curve (up to some modularity issues)}
\end{array}
\]
\[
\text{One if you finite if the period domain is a stack.}
\]

Rule: In certain cases (Deligne), the period domains \( G(\mathbb{Z}) \)
are actual varieties, called Shimura varieties

Ex: Modular curve, \( \frac{\mathbb{H}}{\Gamma} \), or \( \mathbb{A} / \mathbb{Q} \), \( \text{Sp}(2g)/K \), \( \text{Sp}(2g,\mathbb{Z}) \)

Rule: The period map is an injection for curves (Pavlic)

Ex: \text{K3's — the period map is an open immersion.}

Def: If the period map is locally an open embedding, we say the moduli space
\( S \) has canonical coordinates, coming from \( 
\) moduli space, thought
of as a rigid variety.
The moduli space of CY 3s is smooth, and the period map for $H^2$ is locally an open embedding.

Cor: Moduli of CY 3s admit canonical coordinates.