10/6/2014 Joint symp. Seminar

Pseudoconvex

\[ \Sigma \subseteq \mathbb{C}^N. \]

\[ \forall \beta \in \mathbb{C}^N \setminus K \]

\[ \exists \alpha \in \mathbb{P} \text{-polynomial} \]

\[ \text{st. } |P(\alpha)| > \max_{k} |P|. \]

Polynomial hull

\[ K_\alpha := \{ z \in \mathbb{C}^N \mid \forall \beta, \alpha \text{-polynomial, } |P(\alpha)| \leq \max_{k} |P| \} \]

Poly. convex \[ K = K_\alpha. \]

\[ \text{for all } z \text{ see fig.} \]

Holomorphic:

\[ U \text{ open at } \subseteq \mathbb{C}^N \text{ hol. convex if} \]

\[ \forall K \subseteq U \text{, } K_\emptyset(U) \text{ is cpt. too.} \]

\[ U = \Sigma \]

E. Levi: holomorphic \[ \Rightarrow \]

\[ \emptyset \text{ of domains } \subseteq \text{ pseudoconvex.} \]

\[ \Rightarrow \Sigma \cap i\Sigma = \emptyset \text{ rel. submanif.} \]

\[ \psi \text{-convex; } \overline{\text{mean normals converge}} \]

\[ \overline{\text{every}} \text{ complex dimension } \geq 1 \text{ non-negative } \]

\[ \psi \Leftarrow \text{"strict } \psi \text{-convexity"}. \]

\[ \Pi(x) + \Pi(iX) = L(X) \text{ Levi fol.} \]

\[ \mathfrak{d}^* \psi(x) = \mathfrak{d}^* \psi(JX). \]
\[ \partial^2 \Psi = -\omega \Psi \quad \text{skew-symmetric, } \Omega. \]

\[ J - \text{int. form, } \omega \text{ by lie algebra, can construct:} \]

\[ H \Psi = g \Psi - i \omega \Psi \quad (\Leftrightarrow \quad \text{"} \quad \Leftrightarrow \quad \text{"}) \]

\[ \left( \sum_{i} \frac{\partial^2 \Psi}{\partial x_i \partial \bar{z}_j} \right) \]

\[ \Psi \text{ is } J - \text{convex} \quad \text{(called } \text{psf} \text{ for integrable } J) \]

\[ \sum \Leftrightarrow \Psi = C \]

\[ \text{J-convex} \]

\[ \text{J-convexity } \Leftrightarrow \text{(strict) convexity up to biholomorphic change of coordinates.} \]

\[ \text{Non-convex example: annulus in dimension } n > 2. \]

\[ \text{Hartog's extension theorem: Any hol. func. on annulus extends to whole ball} \]

\[ \text{Thus max. principle w.r.t. hol. hull of annulus always contains whole ball.} \]

\[ \Psi \text{ is } J - \text{convex on } C^N \]

\[ (\text{psf, } \Psi \text{ is this way}) \]

\[ \Psi = \Re \Rightarrow \text{holomorphic.} \]

\[ \text{hol. convex domain:} \]

\[ \Psi(x) = \sum (x_i^2 + C) \]

\[ \text{s.t: } \Psi \text{ big on the outside, } \]

\[ \text{but smaller than inside.} \]

\[ \text{We take max}(\Psi, \phi) \text{ in case smooth } \Psi \text{ is } J - \text{convex, global.} \]

\[ \text{(so get J-convex, func. in any way for J-convex domain.)} \]

Look on youtube for yousha draws a cat inside the mouth of a beehive.
Characterization of polynomial & rat'ly convexity:

Then: (Ok) \( \Omega \subseteq \mathbb{C}^n \) — poly. convex (and i-convex!)

\[ \Omega = \{ \mathbf{z} \in \mathbb{C}^n : (\text{defining function}) \} \]

\( \Omega = \{ \mathbf{z} \in \mathbb{C}^n : (\text{defining i-convex function}) \} \)

(global definition)

which extends as an i-convex fun. to \( \mathbb{C}^n \)

(in fact, can do for any defining fun.,)

(in fact, can make this fun. standard at \( \infty \))

\( \Omega \subseteq \mathbb{C}^n \) is rat'ly convex.

\( \omega_\rho = -dd^c \rho \) extends to \( \mathbb{C}^n \) or a kähler symplectic form.

(sympl. form compatible w/ i-convex on \( \mathbb{C}^n \))

(can assume standard at \( \infty \))

(Rmk: polynomial convexity \( \iff \) esthè convexity

(b/c any polynomial can approximate an entire fun. by polynomials)

allows to generalize to other spaces

S rat'ly \( \Rightarrow \) "meromorphic" (ratio of two entire)

When is \( W \subseteq \mathbb{C}^n \) is isotopic to a rat'ly convex domain?

\( n \geq 2 \): the answer is the same as for hoL convexity.

\( n = 1 \): it's yes/no.

\( \mathbb{C}^+ \rightarrow \mathbb{C}^n \) is isotopic to polynomially convex domain

if in addition, \( H_n(W; \mathbb{C}) = 0 \) (and dimensional homology variables)

(sufficiently connected \( \Rightarrow \) huy type \( \leq n-1 \)) (i.e., c.c. contractible; fun. w/ cont. ph. index \( \geq 1 \), still open)
Thus (Nemirovski - Siegel):

\[ W = \mathbb{D}^2 \rightarrow \mathbb{P} \]

\[ W = D_0(x, e) \]

\[ \sum y \]

\[ D_{-\infty}(x, e) \]

Whitney: only level sets of smooth surfaces can be \( \mathbb{R}^4 \). Some special cases for non-removable \( \mathbb{R}^4 \).

(But: for some \( \mathbb{R}^d \) we can put \( S^2 \times \mathbb{D}^2 \) on.

In this case:

\[ D_0(x, e) \]: everything on the fan \( (x, 0) \) \( \neq 2 \)

\[ D_{-\infty}(x, e) \): Whitney list:

\[ e \in \left[ 2x - 4, \infty \right), -2x + 4 + \frac{\sqrt{2} + 1}{\sqrt{2}} \]

+ two exceptions: (1, 2), (0, 0) prohibited

\[ (\text{construct everything not prohibited}) \]

To delete

(Generalize fact that

\[ \mathbb{P} \log \text{ blas } \Rightarrow \mathbb{R}^4 \]

Uses facts: Can find embedding of singular points

"open umbrellas".

Their work implies certain open umbrellas don't exist.

Ingredients: