Penka Georgieva, Enumeration of Real Rational Curves

(w/ A. Zinger)

Q. (C): How many rational curves are there in \( \mathbb{CP}^n \) passing through a given set of constraints? (genus 0).

-Solved using GW theory, by Ruan-Tian, Kontsevich-Maia.

Q. (R): \( \# \) real rational curves ?

(e.g. invariant under involution C-curves)

- Do not consider constraints lying in R-loci, just \( \mathbb{CP}^{2n-4} \) and non-real constraints.

Soln to Q(R) for \( \mathbb{CP}^{2n-4} \) and non-real constraints due to orientability.

To clarify, \( \Phi: M \to M \) anti-symplectic involution.

If \( M^\Phi \neq \emptyset \), its log.

Not considering real pts. w/ constraints on \( M^\Phi \).

First case: degree 1 through 1 non-real point \( = 1 \) such curve.

Will give a relation of the form...
The number of degree $d$ real curves through $k$ points of non-real constants is given by:

$$
\sum \#\text{ real and complex curves of smaller degree } \& \text{ fewer constants}
$$

Only relevant case is:

- $\mathbb{P}^3$: # of degree $d$ real curves through a $d$ points $N_d^{\text{R}}$.

Relation is:

$$
N_d = \sum_{-2d_1 + d_2 = d} \binom{d-2}{d_2-1} \binom{d-1}{d_1} N_{d_1}^{\text{C}} N_{d_2}^{\text{R}}
$$

For $d_1 \geq 1$

- $d = 1 \rightarrow 1$
- $d = 3 \rightarrow -1$
- $d = 5 \rightarrow 5$
- $d = 7 \rightarrow -85$

(don't always alternate sign)

- These #s vanish in all other cases.

- Every time we use $\phi \cdot \alpha = \alpha \Rightarrow$ automorphism.

Recall: (complex case): $(M, \omega)$ symplectic.

$$
W_k(M, \omega) \xrightarrow{\text{ev}} M^k
$$

$\text{Diff} \bar{M}_k$ DM space.
If \( \eta = (q_1, \ldots, q_k) \in H^*(M) \), then

\[
\langle \eta_1, \ldots, \eta_k \rangle_{g} = GW_k(M) := \int eV^* \eta_{\bar{M}_k(M, f)} \]

\[
= \# \left\{ [u] \in eV^2 \left( \mathcal{PD}(\eta) \right) \right\}
\]

\[\uparrow\]

known that count is pure if chosen of complex representation

\[\overline{M}_4 = S^2 \cup 3 \text{ special points.}\]

\[\uparrow\]

gives any \( M \), makes pt.

\[
\int eV^* \eta \cup f^* (\mathcal{PD}(pt))
\]

[\overline{M}_k(M, f)]

\[\uparrow\]

\[\ldots\]

\[\uparrow\]

\[\mathrm{For \ CP^2, \ count}\]

\[\Sigma \mathrm{CP}^2 \]

\[= \Sigma \mathrm{PL}(\text{Kuranishi family})\]

\[\text{Gives measure, \ the only shifts can be } 0.\]
Setup: $(M, \omega, \Phi)$ \hspace{1cm} $\Phi: M \to M$

$\Phi^* \omega = -\omega$, \hspace{1cm} $\Phi^2 = \text{id}$

$E_k: \mathbb{C}P^{2n-1}$ to natural involutions:

* $T_k: [z_1, \ldots, z_n] \to [\overline{z}_1, \ldots, \overline{z}_n] \to \text{fix} = \mathbb{R}P^{2n-1}$

* $Z_k: [z_1, \ldots, z_n] \to [-\overline{z}_1, \overline{z}_2, \ldots, -\overline{z}_n, \overline{z}_{n-1}]

\downarrow \text{no fixed locus.}

$\tau = \tau_1$ \hspace{1cm} $\mathbb{C}P^1 \times \overline{\mathbb{C}P^1}$ represent the two topological involutions

reflect \hspace{1cm} antipodal map.

For $c = \tau$ or $\eta$, \hspace{1cm} $\overline{M}_{\Phi, c}^k(M, b) = \left\{ u: \begin{array}{c} \text{closed} \subset \to M \end{array} \begin{array}{c} \overline{\Phi} \end{array} \left\{ \begin{array}{c} \Phi \circ u = 0, \vspace{1cm} \Phi \circ u \circ c = u \end{array} \right. \right\}

\begin{array}{c} \vspace{1cm} \text{k pairs of adj. constancy} \vspace{1cm} \{(z_i, \overline{z}_i), -\overline{(z_i, \overline{z}_i)}\} \end{array}$

Under certain topological conditions, can solve a couple issues:

* these are orientable

rest
If \( c = 2 \), then no boundary, odd degree curves in \( \mathbb{P}^{2n-1} \)

\( \Rightarrow \) no boundary \( \Rightarrow \) excludes 2.

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\( \overline{\mathcal{M}} \) has fixed locus, \( + \) \[ \begin{array}{c} \overline{\mathcal{M}} \cup \overline{\Phi_{12}} \Rightarrow \text{count various in all known cases.} \\ \text{degree even,} \end{array} \]

From now on, use the notation \( \overline{\mathcal{M}}_{k}^{\Phi_{12}}(H, b) \) for which even of these cases we're in.

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For \( q_2, -r_k \in H^*(M) \),

\[ \langle q_2, -r_k \rangle_k \overline{\Phi_{12}} = \text{RGW}_{k}^{b}(q_2, -r_k) = \int_{\overline{\mathcal{M}}_{k}^{\Phi_{12}}(H, b)} [M] \]
Two ways to obtain a relation

\[ M_4, \quad \text{pt.} \]

(a) use \[ \Rightarrow M_{\mathcal{a}}(\mathfrak{c}) \]

(b) use \[ \Rightarrow M_3, \quad 1\text{-dimensional homology class} \]

\[ \Rightarrow \quad \text{real, real.} \quad \text{either way, code, 2.} \]

Let \[ \overline{M}_k(b, \gamma) = \left\{ (\eta) \in M_k^\Sigma (b, \gamma) \mid \eta \in \text{ev}^{-1}(PD(\eta)) \right\} \]

\[ \Rightarrow 2\text{-dim.} \quad \text{and } k > 3 \]

\[ f : \overline{M}_k(b, \gamma) \rightarrow \overline{M}_4 \]

\[ \left\{ \eta : \begin{array}{c} 1 \rightarrow M_3 \rightarrow \overline{M}_3 \end{array} \right\} \quad \Rightarrow \quad \overline{M}_4 \]

Look at \[ f^{-1} \left( \begin{array}{c} 1 2 \\ 3 \end{array} \right) = \begin{array}{c} 1 \frac{1}{2} \\ \frac{1}{3} \end{array} \]

\[ \text{not cyclic value,} \]

\[ \text{style read maps.} \]

\[ \text{not j.} \]

\[ \text{style (c.e.)} \]

\[ \text{sym.} \]
So get a relation:

\[
\begin{array}{c}
\frac{12}{3} \quad 3 \quad 12 \\
\frac{13}{3} \quad 2 \quad 2 \\
\frac{2}{2} \quad 13 \\
\frac{12}{2} \quad 3 \quad 12
\end{array}
\]

\[\overline{M} = \frac{13}{2} \]

\[\overline{M}_2 \]

\[\text{boundary spheres} \]

\[\text{the 4 lines connect} \]

\[\text{are precisely these} \]

Q: is there an interpretation of, say an associated relation?
\( \Lambda : C \overset{\text{define}}{\rightarrow} R \overset{\text{and}}{\rightarrow} Q \overset{\text{and}}{\rightarrow} Q \)

This relation tells us that:

\[ R_{\Phi}^1 M_1 \star M_2 := y_1 \star R_{\Phi}^2 y_2 \]

Real constructs: moduli spaces are not oreable; real classes of trusted code; also ugly are these constants.