Above:
\[ \text{closed, smooth, Riemannian, inj} = 1 \, . \]

\[ T^*Q \to \text{tangent} \]
\[ (\text{moduli}) \quad \text{Obst.} \]
\[ TQ \]
\[ T^*Q \to R \]
\[ |p| \]

\[ \text{Hamiltonian flow of } |p|^2 \quad \text{on } T^*Q \quad \xrightarrow{\text{flow}} \quad \mathcal{H}(T^*Q) \]
\[ \text{geodesic flow on } TQ \quad \xrightarrow{\text{flow}} \quad H^*(\mathcal{L}Q) \]

True only if \( Q \) is spin (T.Kragh)!

\( \bullet \) Let \( |Q| \) be an orientation line of \( Q \) (local system of \( \mathbb{Q} \)),
set by pull back under \( ev_0 : LQ \to Q \), becomes a local system on \( LQ \).

\( \bullet \) Let \( \sigma \) be the local system on \( LQ \) classified by the transgression of
\[ w_2(Q) \in H^2(Q, \mathbb{Z}_2) \to H^3(LQ, \mathbb{Z}_2) \]
(homology: \( H_1(LQ) \to H_2(Q) \)),
"sweep"m

\( \bullet \) Let \( w_1 : LQ \to \mathbb{Z}_2 \) be \( \sigma \) on orientable loops \( \sigma \)
1 on non-orientable loops,
\[ \mathcal{L} : Q : = w_1^{-1}(1), \ i.e. \{0,1\} \]
Def: \[ n : = \sigma \otimes |Q|^{1-w_1} \]
i.e. \( \sigma \otimes |Q| \) on \( L^0Q \), \( n \) on \( L^1Q \),
local sys. on \( LQ \).
Thm: (A): \[ H_{n-k}^*(\mathcal{L}Q, \eta) \cong SH^* (T^* Q) \] 

Remarks:  
1. [Note: missing content] 
2. Can twist everything by an extra local system \( \nu \) on \( \mathcal{L}Q \). Then, \[ H_{n-k}^*(\mathcal{L}Q, \eta \otimes \nu) \cong SH^* (T^* Q, \nu) \]. 
3. Usual grading is by "CZ index." Here, use 
   \[ \deg(x) = n - CZ(x) + \left[ w_1(x) \right] \] 
   cohomological. 
   \[ q : T^* Q \to Q \] 
   \[ \text{CZ} \times \text{loop in } T^* Q; w_1(x) = w_1(q(x)) \]. 

Operations: \[ \text{Assume } \nu \text{ is transgressed from } H^2(Q, \mathbb{Z}_2) \]. 

Then: A natural \( \mathbb{Z} \)-graded twisted BV structure on 
   \[ SH^*(T^* Q, \nu) \] and \[ H_{n-k}^*(\mathcal{L}Q, \eta \otimes \nu) \], 
   and the map intertwines the structure.

\[ \Delta \text{ operator of degree } -1 \text{ with square } 0. \]

BV \[ \ast \text{ multiplication (grade) associative & commutative).} \]
on $H_x LQ$ by "strong product."

$H_x (LQ)$ has no BV st. if $Q$ is not countable

$\sim \sigma \otimes |Q|$ an evritable component

But because we use

$\sim \sigma$ on $L^1 Q$, associatively holds up to

$\text{sign } \deg(a) \pm w(a)$. $a/\text{coalg}$

If we consider $SH^*(T^*Q)$ as a $Z_2$ graded vector space,
then it has no honest BV structure, always.

Can only get this in $H_x LQ$ by shifting evrtable $\text{sign evrtable layers}$

**Questions:**
1. What is the categorical interpretation??

   Suggests in non-total case, $\text{Hox } = H^*$, but

   not well (still get a $Z_2$-graded) BV, just a

   $Z_2$-graded BV (unshrink).

2. What is the spectrum -level refinement?

   We have many $(H_{x-1}(LQ), \sigma, BV)$ BV structures.
4 different proofs

1. Virtues: couple $SH^*(T^*Q)$ by "finitely-dual approximation.

   (generating function) \rightarrow

   Relate to homology $Z^n Q \subset Q^n$

   \uparrow

   pts s.t. success may be $\text{inj}(Q)$ apart.

   \begin{align*}
   \cup_{n \geq 1} Z^n Q & \subset LQ \quad \text{n.e.} \\
   \end{align*}

2. Abbondandolo-Schwarz:

   Define $H^*_L(Q)$ using a $W^{1,2}_{R,\text{subsol}}$ model for loop space,

   + More theory of energy functional: $E =$ length

   (Define: smoothness of time flow is merely better than $C^2$)

   They prove that for careful choice of Floer data ($f \neq \epsilon$),

   have chain map

   $CM^*(E) \rightarrow \mathcal{C}^d(T^*Q)$

   - Can be rep. by chain & plx.

3. Salamon-Weber: degenerate Floer $\epsilon \rightarrow Q$ so that

   $\exists$ bijective correspondence between hol. cylinders $\mathcal{B}$

   flowers in a $L^2$ model for $LQ$.

Lot of careful & delicate analysis.

(N.B. this gives more information could hope to look for "related invariance of chain complexes, i.e. Whitehead torsion")
4. Family Floer homology:

Map $H_c(LQ) \to SH(T^*Q)$.

Let's say have $P \to LQ$, so $\delta_p : S^1 \to LQ$ for $p \in P$.

Smooth manifold w/ boundary (compact)

Can assume $d(\delta_p(0), \delta_p(\Theta + \frac{1}{r})) < inj(Q)$ for same single $r$ independent of $p \in P$.

Send $p \to \bigcup T^*_{\delta_i} Q$, where $\delta_i = \delta_p \left( \frac{i}{r} \right)$

Get a family of Lagrangians over $P$; don't intersect.

Consider $H : T^*Q \to \mathbb{R}$

$(q,v)$

$H = h(tu)$

Claim: $\exists$ a unique intersection point between $T^*Q$ and $\phi_t(\cap_{l=1}^n)$ if $\phi$ is Hamiltonian flow of $H$. 
The map:

\[ T^* Q \times T^* Q \rightarrow C^+ (P \times Q) \]

"covering element" of \( C^+ \) defines the desired map.

\[ \text{preimage converges to orbit of Hamiltonian } H \rightarrow \mathbb{C}^* (L^2) \downarrow \rightarrow \mathcal{L}^* (T^* Q). \]

Reverse the process:

Given a generator \( x \) of \( \mathcal{L}^* (T^* Q) \),

consider the moduli space

\[ \text{var freely.} \]

Warning: Why does the distance between \( q \) and \( q' \) stay less than \( \text{inj} (Q) \)?

The main idea is a version of the maximum principle for hol. curves due to (A.-S.).
The closer away the fiber lies, the higher up the intersection point it is.

(If fiber close, intersection close to zero section).

Maximum principle: can ensure limit on least height of hol. curve.

If inputs lie below $|w|$, all outputs do too.

Can check: these maps are actually inverse on homology, essentially construct using same techniques.