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(joint w. John McCleary)

(Exercise: show manifold $V_2(\mathbb{R}^{2n+1})$ (sphere bundle over $T S^{2n}$).

Prove it has only many geodesics!)

Old problem: Does a Riemannian manifold have an $\infty$ # of closed geodesics?

(Morse theory on $LM$ (free loop space of $M$). \( \phi \) maps $\mathbb{R}/\mathbb{Z} \to M$.

Gromoll-Meyer: Suppose $M$ is s.c. & \( \{ b_i(LM,F) \} \) is unbounded (for some field $F$). Then, any metric on $M$ has an $\infty$ # of closed geodesics.

Sullivan, Vaug-Poirier: Suppose $\dim Q H^\ast(M,\mathbb{Q}) \geq 2$.

Then, $\{ b_i(LM,\mathbb{Q}) \}$ is indeed unbounded. (app. of cat's hpy theory).

\textbf{Btw. Gneiss metric:}

Morse-Theory: $E:LM \to \mathbb{R}$, $E(x) = \int_0^{\delta_n} g_\delta(\dot{x}(t),\dot{x}(t)) dt$.

A closed geodesic (parametrized by arc-length) critical point, as is $\delta_n$, where $\delta_n(t) = x(nt)$. Multiplicity:

(Thus is why need $b_i \to \infty$.)

If $\pi_1$ finite, apply to universal cover.

If $\pi_1$ infinite; # of isometry classes; take one & rep. in each class.

\textbf{McCarther:} Suppose $X$ is a finite CW cplx. (simply connected); and $\dim Q (H^\ast(X,F)) \geq 2$. Then, $\{ b_i(S^X,\mathbb{F}) \}$ is unbounded. (What about $EX$? ?)

\textbf{McCarther} (1987) study this for the usual symmetric spaces.
String homology: (Chas-Sullivan).

\[ M \mapsto HL_{x}(M) \text{ string homology ring} \]

with property
\[ HL_{x+d}(M) = H_{*}(LM) \quad \text{if } d = \dim M. \]

\( HL_{x}(M) \) graded commutative; finitely generated.

Therefore \( \beta_{*}(HL_{x}(M)) \) is unbounded \( \iff \) \( HL_{x}(M) \) contains a poly. algebra on two variables.

N.B. \( H_{*}(LM, F) \) is a ring, but not a general graded commutative, so doesn't apply.

String product:

\[ H_{s}(LM) \times H_{t}(LM) \to H_{s+t-d}(LM) \]

Say family of loops \( \mathbb{S}^{1} \times Y \)

\( \mathbb{S}^{1} \times Y \to M \subseteq Q \)

\[ \mathbb{S}^{1} \times Y^{*} \]

Assuming some transversality, etc., can make this work.

Cohen-J: Formalize using methods of homotopy theory into string product

\[ HL_{p}(M) \times HL_{q}(M) \to HL_{p+q}(M), \]
To first approximate, this looks like

\[ H^*(M) \otimes H_*(SM) \]

\[ \xrightarrow{\text{intersect}} \quad \xrightarrow{\text{compose loops}} \]

**CJY:** Construct a spectral sequence

\[ E^r_{s+t}, d^r: E^r_{s+t} \rightarrow E^r_{s-r, t+r-1} \]

\[ E^2_{-a,b} = H^a(H) \otimes H_b(S^2M) \] (working over field \( F \)).

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\[ H^0M \rightarrow H^2M \rightarrow H^1M \rightarrow H^0H \]

\[ \xrightarrow{\text{stops at 2, so finite # of obstructions + lifting}} \]

\[ H_0SM \rightarrow H_0LM. \]

**N.B.:**

\[ H_*(SM) \rightarrow H_*(SM) \] edge homomorphism of spectral sequence.

**N.B.** Not sure special sequence. (Involves mixed homology & cohomology)

(b) Really uses manifold structure on \( X \), just fancier Duality.
N.B.: \( H^k LM \) and \( H^k(\Sigma M) \) are very different.

Typically this is pretty trivial products, but this may have different non-trivial (e.g. poly products).

Fix prime \( p = 2 \). Work over \( \mathbb{Z}/2 \).

Results derived from the CSY spectral sequence.

**Theorem 1:** There exists an integer \( N \) (power of 2 \( \equiv p \) with following properties:

(a) If \( x \in H^k(\Sigma M) \), then \( x^N \) is in the image of \( F \).

(b) If \( x, y \in H^k(\Sigma M) \), then \( x^N \) and \( y^N \) (graded) commute.

(LPC)

**Argument for (a):** Simple:

\[ d_2(x) \text{ may not equal } d_2(0) \text{, but} \]

\[ d_2(x^2) = 2d_2(x) = 0 \text{ so } \]

\[ x^2 \text{ survives } \rightarrow \text{ page } 3 \text{. Repeat.} \]

(b) follows.

**Theorem 2:** \( H^k(\Sigma M) \) carries a polynomial algebra on two generators

iff \( H^k(\Sigma M) \) does. (Uses Theorem 1).

(Really relies on manifold structure!)

Apply to Stiefel manifolds:

\[ V_n = V_2(\mathbb{R}^{2n+1}) \]

\[ H^*(\mathbb{R}^n), \mathbb{Z}/2 = \mathbb{Z}[u, v] \text{ with } |u| = 2n-1, |v| = 2n, V_n = V_2(\mathbb{R}^{2n+1}). \]

(cohomology is graded polynomial.)
Hopf Algebra Techniques

- We know \( H_c(2M) \) is doubly infinite (meaning by \( \to \infty \)).
  if \( \dim (Q H^*(M)) \neq 2 \).
- \( H_c(2M) \) satisfies LPC condition.
- \( H_c(2M) \) is a connected Hopf algebra, co-commutative Hopf algebra.

A Hopf algebra \( H \) is solvable if \( \exists \) a finite filtration:

\[
0 = A[0] \subset A[1] \subset \cdots \subset A[s] \subset A[\infty] = H
\]
by normal sub-Hopf algebras, (i.e. quotient is Hopf alg. too)
such that

\[
\]
is an abelian Hopf algebra. (comm. & cocomm.),
(i.e. analogous to fr. iter. extension by abelian gps.)

Theorem: Suppose \( H \) is a connected, finitely generated commutative solvable Hopf algebra, satisfying LPC. Then, \( H \) is doubly infinite

\[\iff \] \( H \) contains a polynomial algebra on two generators.

Two tricks for this:

* Milnor-Moore: \( A \otimes B \) Hopf algebras; \( C = F A \otimes B = B / A \).
  \( B \cong A \otimes C \) as \( A \)-modules & \( C \) comodules.
Use this iteratively (+ doubly infinite rnkhe) to construct a surjective
\[ \Gamma \rightarrow F[x,y] \]

Pick \( v \rightarrow x \)

Then \( u, v \) commute, map onto \([x^w, y^w]\),

so \( v \).

Claim this seems to explain exactly \text{in McCleary--Ziller!}

(should be that \( \Gamma \) solvable).

Factual:
- \( A \) Hopf algebra, s.t. all \( x \in A \) have finite height.
  \( (\text{i.e. } x^n = 0 \text{, s.o.e } x) \).

Is \( A \) finite? (Don't know this)

(Recall Burnside problem: doesn't work for graphs)

Known that \( H_n(\Omega M) \text{ are very special Hopf alg} \).

(Feit, Halpern, --)

Is it possible that this is enough to finish off the problem??

Non-s.c. case? Generally Hurewicz may not apply??

Q: Does \( H_* M \) always have fin. rank? (Known for s.c.)
(Dunno known by Scarf).