Mclean, On the Symplectic Invariance of log Kodaira Dimension.

$X$ projective variety, study there up to birational isomorphism.

- Numerical properties of the canonical bundle $K_X$.
  - e.g., $H^0(X, mK_X)$.
  - More simply, $P_m := \text{rank} \left( \frac{H^0(X, mK_X)}{\text{sections of } K_X^m} \right)$, $m$-th plurigenus.

Even more coarsely, look at rate of growth:

$$\text{Kodaira dimension} := \limsup_{m \to \infty} \frac{\log P_m}{\log m} \in \{-\infty, 0, \ldots, \dim X\}$$

If $P_m$ grows like a poly. of degree $k$, this number is $k$.
  - if $0$, then $-\infty$.

Examples:

- $\mathbb{P}^1$:
  - $k(\mathbb{P}^1) = -\infty$

- $E$:
  - $K(E) = 0$

- Higher genus:
  - $K(C) = 1$.

Related also to curve $C$: positive, zero, negative:

- $k$ (low deg. hypersurfaces in $\mathbb{P}^n$) = $-\infty$

- $k$ (high degree) = $n - 1$.

- Study rational curves on $X$.
  - Uniruled varieties: $\exists$ a non-constant map from $\mathbb{P}^1$ to each pt. in $X$. 
Globally connected varieties: \exists \text{ a map from a rect' curve joining any 2 pts in } X.

\rightarrow \text{ study these from a topological/symplectic pt. of view.}

e.g., \dim X > 2 \text{ \ 
Donaldson theory \ 
SW theory} \quad K_X \text{ deformation invariant, } K(X) = 2.

(Witten '84)

\text{Poincare are deformation invariant}

(Friedman, Morgan '97).

\underline{Gromov-Witten theory}. \quad \underline{Kollar-Ruan: uniruledness is an sympl. invariant.}

\text{natural connectedness is a symplectic invariant, if } \dim X \leq 3.

(Voisin '08, Z. Tian '10).

\rightarrow \text{ less is done for open varieties.}

\rightarrow \text{ less is done in higher dimensions.}

\underline{Smooth affine varieties}

\[ A \longrightarrow \mathbb{C}^N \quad \omega_A := \text{cst div} \sum_{i,j} \frac{dx_i \wedge dy_j}{x_i \wedge y_j} \]

This is an invariant up to modulus (Euler characteristic)

(Gromov)

Define log \text{ kappa dimension of } A \text{ by } X \text{ (an alg. variety).} \quad D = X \setminus A = \text{ union of transversely intersecting hypersurfaces}

\text{Then A). Let } A, B \text{ be symplectomorphic affine varieties.}

1) If \dim A > 2, \quad H^*_X (A, Z) = H^*_X (pt, Z) \text{ then}

\[ \bar{K}(A) = \bar{K}(B). \]

Define \( \bar{K}(A) \) (by log dim.)

\[ \text{len sup} \log \text{ rank } H^0(X, k_X + D) \]

\text{Claim: if } A, B \text{ symplectomorphic affine varieties, then } C \times A \text{ is symplectomorphic to } C \times B.
(1i) if dim $A = 3 + \text{technical conditions}$

then $\mathbb{R}(A) = 2 \Rightarrow \mathbb{R}(B) \leq 2$.

**Theorem 3:** If $P$ and $Q$ are projective, and $A \leq P$ and $B \leq Q$ are symplectomorphic open affine subsets, then $P$ is uniruled if $Q$ is uniruled.

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**M. Lavelle domain**

**Θ = 1 form**

(1) $d\Theta = \omega$ symplectic

(2) $\chi_{\omega}$ vanishes along $\partial M$

Dual of $\Theta$. 

$M$ is $(k, \Lambda)$ uniruled if:

- $p \in M$, an open and a symplectic almost complex structures $J$ on $M$ with $\Theta \circ J = dr$ at near $\partial M$ where $r$ is a function whose highest level set is $\partial M$, open Riemann surface.

Then: A proper holomorphic map $u : S \to M$ branched passing through $p$ with

- $\mathcal{O}$ $\text{genus 0}$ ($\text{top} \ to \ \omega \leq k-1$ cycles?)

- $|H_1(S, \mathbb{Q})| \leq k-1$

- $\int u^* \omega \leq \Lambda$

Complete $M = M \cup \partial M \times [0, \infty)$

**Lemma:** If $M \cong S$ then if $M$ is $(k, \Lambda)$ uniruled, $F \cong L$ uniruled if $N \cong (k, \Lambda')$ uniruled.
\( A \hookrightarrow \mathbb{C}^N \)

affine variety

\[
\begin{align*}
\text{Define} \quad A &:= A \cap \text{very large ball in } \mathbb{C}^N \\
\Theta_A &:= \sum_j r_j^2 \partial \partial^* z_j \quad |A| \\
\hat{z}_j &= r_j e^{i\theta_j}
\end{align*}
\]

This is a Liouville domain.

Another theorem is that \( A \sim \hat{A} \).

Defn: \( A \) is \( k \)-uniruled if \((k, x) \) uniruled for some \( x \).

Def 2: \( A \) is algebraically \( k \)-uniruled if through any \( P \) \( f: \mathbb{P}^1 \setminus \{ \text{at most } k \text{ points} \} \to A \) passing through \( P \).

Thm: If \( A \) is \( k \)-uniruled then it is algebraically \( k \)-uniruled.

Idea of proof: use degeneration to normal cone ("alg. neck stretching") + compactness result by Joel Fish.

Diagram:

\[
\begin{align*}
\text{non-zero fibers are isomorphic to } A \\
\text{central fiber } &= X \times E \\
\text{constriction of } A.
\end{align*}
\]

Choose compact subset \( \hat{A} \) (to get finite value), pull \( A \) into \( \hat{A} \).

Push to \( A \), restrict to \( A \), push into central fiber, get some compact piece in \( X \).
You have a curve passing through each point in every non-zero fiber (hence domain)

\[ \Rightarrow \text{ (by compactness)} \]

Each curve passing through every point is the central fiber with a compact component mapping to X.

(One part restarts \( A \) still has right \( H \) \( \rightarrow \) technical arg, maximum principle, etc.)

**Proof of Theorem H (ii):** \( \dim \mathbb{C} A = 3 \), + technical conditions.

\[ \overline{k}(A) = 2 \Rightarrow \overline{k}(B) \leq 2. \]

(Kodaira) \( \Rightarrow \) A has a \( C^1 \) fibration.
compactify A to X, extend to a \( P^1 \)-fibration.

Use GW theory on X to show that \( A \) is (2, 3) uniruled.

(\( GW \) with corresponds to class of \( P^1 \) fiber in X).

This means \( A \) is 2 uniruled.

So by our lemma, \( B \) is 2-uniruled.

\[ \Rightarrow B \text{ algebraically 2 uniruled} \]

Our theorem

\[ \Rightarrow \overline{k}(B) \leq 2. \]

[I: take]

sub result.