Def: A log scheme is a scheme $X = / $, a sheaf of monoids $M_X$ and a monoid homomorphism

$$ \alpha_X : M_X \to \mathcal{O}_X^{*} $$

monoid w.r.t. multiplication,

$$ \alpha_X(p+q) = \alpha_X(p) \cdot \alpha_X(q) $$

s.t. $\alpha_X : \alpha_X^{-1}(\mathcal{O}_X^{*}) \to \mathcal{O}_X^{*}$ is an isomorphism (of sheaves).

A morphism of log schemes $f : (X, M_X) \to (Y, M_Y)$ is a map morphism $f : X \to Y$ and a map

$$ f^\# : f^{-1}M_Y \to M_X \text{ s.t. } f^\# \text{ commutes.} $$

Let $M_{(X,D)} = (j_*(\mathcal{O}_X^{*} \cap D)) \cap \mathcal{O}_X \xrightarrow{\alpha_x} \mathcal{O}_X$

Def: A pre-log structure is a monoid hom.

$$ \alpha : M_X \to \mathcal{O}_X^{*} \text{ (no condition on } M_X). $$

The associated log structure is

$$ (M_X \otimes \mathcal{O}_X^{*}) / \{ (p, \alpha(p)^{-1}) \mid p \in \alpha^{-1}(\mathcal{O}_X^{*}) \} $$

w/ map $(\alpha_X \otimes \text{inclusion}) \to \mathcal{O}_X^{*}$.

E.g. Let $P$ be a monoid, $(\sigma \subseteq \mathbb{R}^n \text{ a rational polyhedral cone, } P = \sigma \cap \mathbb{Z}^n)$

Pick

$$ \sigma : \mathbb{R} \to \mathbf{Spec} k[P] \xrightarrow{\text{red}} P \to \mathcal{O}_{\text{Spec } k[P]} \xrightarrow{\alpha} \text{associated log structure} $$
$X = \text{Spec } k[P]$

$\partial X =$ complement of big toric orbit

$\sim$ associated log structure above $= \mathcal{M}(X, \partial X)$.

More generally, the log structure associated to a map

$$
\begin{array}{c}
\mathcal{P} \\
\downarrow \\
X \\
\downarrow \\
\text{Spec } \mathbb{Z}[P]
\end{array}
$$

is called a fine log structure

A log structure is fine if (locally it can be described in this way on some étale open cover).

(N.B. You can always pull-back a log structure).

Log geometry allows you to treat certain varieties that look very singular as smooth.

Let's take EGA def of smooth replace "scheme" $\rightsquigarrow$ "log scheme".

A log smooth morphism locally looks like

$$
\begin{array}{c}
\text{Spec } \mathbb{Z}[P] \\
\downarrow \text{ induced by} \\
\text{Spec } \mathbb{Z}[Q]
\end{array}
$$

$e.g.$ $\mathbb{Z}[\mathbb{N}]$ natural #'s (include 0).

$\text{Spec } \mathbb{Z}[P]$ $p \in P$

\[\text{Spec } \mathbb{Z}[N] \quad \text{ by } \quad \text{inj} \quad (\text{is general new finite kernel})

\text{Spec } \mathbb{Z}[P]$

$e.g. \quad Y = \text{Spec } k \otimes_{\mathbb{Z}[N]} \mathbb{Z}[P] \quad \text{Spec } \mathbb{Z}[P]$

\[\text{Spec } \mathbb{Z}[N] \quad 1 \in \mathbb{N}

\text{Typically as an orbifold scheme, } Y \text{ is very singular (e.g. normal crossings)}$

\rightarrow \text{but here, } Y \text{ becomes smooth}
Excuse me, what is a map \( \text{Spec } k^+ \) into a variety?

If \( f: X^+ \to Y^+ \) is log smooth, then

\[
\mathcal{S}^1_{X^+/Y^+} = \mathcal{S}^1_{X/Y} \oplus (\mathcal{O}_X \otimes \mathcal{M}^p_X)
\]

where \( X \) is generated as an \( \mathcal{O}_X \) module by

\[
d\alpha_X(m) = \alpha_X(m) \otimes m \quad \text{[\(\alpha_X(m)\) is \(\text{type}\) of \(\text{dlog}\) map? ]}
\]

\[
(0, 1 \otimes \mathcal{M}^p_X(m))
\]

i.e., if you pull something back for \( Y \), \( \text{dlog} \) of it is \( 0 \).

\[\text{e.g. } D \subseteq X \text{ a normal crossings divisor,} \]

\[X^+ = (X, \mathcal{M}(x, D)) \text{ trivial log structure, i.e. } \mathcal{M} = \mathcal{O}_X \]

\[Y^+ = (\text{Spec } k, k^+) \]

then \( \mathcal{S}^1_{X^+/Y^+} \) is sheaf of 1-forms on \( X \) with logarithmic poles along \( D \).

(Assumption log smooth makes (*)) locally free (not obvious?)

\( \mathcal{S}^1_{X^+/Y^+} \) is locally free if \( f \) is log smooth.

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Let \( X \) be a normal crossings variety.

When do we have \( X^+ \to \text{Spec } k^+ \) log smooth.

(Konno-Namikawa answered this)

Define \( N_D = \text{Ext}^1_{\mathcal{O}_X}(\mathcal{S}^2_{X/k}, \mathcal{O}_X) \)

This is a line bundle on \( D = \text{Sing} (X) \) (local computation). [requires normal crossings]

Then giving \( X^+ \to \text{Spec } k^+ \) log smooth

\[\iff \text{for } f^*(X, N_D) \text{ nowhere vanishing.}
\]

(\( \text{so need } N_D \cong \mathcal{O}_D \).
Let $\Sigma$ be an edge with $X_{\Sigma} \subseteq X$ the corresponding 1-dimensional stratum.

Then $\Sigma$ has a singularity w/ monodromy $(1 \ 0 \ d \Sigma)$, $N_{D_\Sigma} \cong \mathbb{C}^2$, i.e., not good!

So no non-vanishing section if $d \Sigma > 0$. There are at least some sections!

(If $d \Sigma < 0$, no way to put a complex structure on 2-torus bundle.)

Choose a section $f \in \Gamma(X, \mathcal{N}_D)$ which doesn't vanish on an entire $X_{\Sigma}$

(it will vanish probably at $d \Sigma$ points, maybe w/ multiplicities).

Let $Z = \{z = f\}$ be the locus of $f$.

Get a $(X \setminus Z)^+ \to \text{Spec } k^+$ log smooth, log smooth (\text{\textasteriskcentered} X \text{\textasteriskcentered})

(Main reason Gross-Siebert papers are so long: it's all log smooth justice doesn't include non-free case, which arises if one tries to extend (\text{\textasteriskcentered} X \text{\textasteriskcentered}) to all of $X$)

In particular, if one extends to $X$, resulting log structure is not fine at $Z$.

$j : X \setminus Z \to X$ the inclusion.

Define: $\Omega^1_{X/\mathbb{C}} = j^* \Omega^1_{(X \setminus Z)^+/\mathbb{C}}$

Q: What's still at special points?
Want to compute $H^*(X, \mathbb{R}^1 \times \mathbb{R}^t)$

Crucial point: $v$ a vertex

Let $X_v = X_v \setminus (\mathbb{Z} \cap X_v) \hookrightarrow X_v$

Then $X_v \times (\mathbb{R}^t / k^t \mid X_v \setminus (\mathbb{Z} \cap X_v))$ is a trivial rank 2 vector bundle.

Glue these:

Glue: sets of lattice

$\log n$ the log

Here $f_w = f \mid X_w$

$\log n$ the log

Looks like monodromy.

Why did we want to compute $H^*(X, \mathbb{R}^1 \times \mathbb{R}^t)$? Because this is the right guy to consider in big site of smoothness.

$X_v = X_v \times (\mathbb{R}^t / k^t \mid X_v \setminus \mathbb{Z})$. $H^0 \hookrightarrow \mathcal{L}_v \otimes \mathbb{C}_k$

For $i \leq t - 1$, $\mathbb{R}^i \mathcal{L} = \mathbb{R}^{i+1} \times k^t \mid X_v \setminus \mathbb{Z}$. For $i = t$, $\mathcal{L}$ turns out to be line. (may be a line by more)

If $\dim \mathcal{L} = 2$, $\mathbb{R}^1 \mathcal{L} = \mathbb{R}^2 \times k^t \mid X_v \setminus \mathbb{Z}$. $\mathcal{L}$ should be flat at $x \in \mathcal{L}$. $H^0 \hookrightarrow \mathcal{L}_v \otimes \mathbb{C}_k$

monodromy-invar.

Section of $\mathcal{L}$ in a neighborhood of $\mathcal{Z}$. $x \in \mathcal{Z}$, trivial fiber on a point.
\[ \xi^* k = \bigoplus \sigma \xi_0 \bigg/ x \sigma_k, \quad \text{given } x \in \xi^* k, \]

\[
(\text{clc.}) \xi_0 \supseteq \cdots \supseteq \xi_{k+1} = \bigoplus \xi_0 \delta_i - \cdots - \xi_{k+2} - \xi_i - \cdots - \xi_k + \sum_{i=1}^{k+1} \xi_0 \delta_i - \cdots - \xi_{k+2} - \xi_i - \cdots - \xi_k, \]

\[ + (-1)^{k+1} \xi_0 \delta_0 - \cdots - \xi_k \bigg/ x \sigma_{k+2}. \]

Check: \[ \xymatrix{ 0 \ar[r] & \Omega^1_{X^+ / k^t} \ar[r] & C^* \ar[r] & 0 } \]

is a resolution.

"Ogus" gives enough, yet exact seq.

\[ \xymatrix{ 0 \ar[r] & \Omega^1_{Z^+ / k^t} \ar[r] & \Omega^1_{Z \times \Delta} \ar[r] & 0 } \]

\[ \forall \sigma \in \Sigma. \]

could be higher cohomology if \# \Sigma Z \times \Delta > 1.

If simple \( B \# \Sigma Z \times \Delta = 1 \), then \( \sigma \subseteq H^2 \).

That do \( \sigma \theta \beta \) kills monodromy w.r.t. shift, \( \sigma \theta \beta / \Sigma^1 \) is

\[ \Rightarrow \]

\[ \Gamma(X, C^*) \cong \check{C} \text{ech complex for } \Delta \text{ on an abelian open cover of } B. \]

\[ \therefore \quad H^*(X, \Sigma Z \times \Delta^+ / k^t) = H^2(B, \sigma \times \Lambda \otimes k) \]

if \( i : B \setminus \Delta \rightarrow B \).

Don't do smoothing \( X \rightarrow X \) is a general fiber.

\[ \text{GL}(B) \times R \text{ (ax)} \]

\[ \text{Spec } k[C(t)] \quad \xymatrix{ \text{Spec } k[C(t)] \ar[r] & \text{Spec } k[C(t)] } \]

\[ \text{Rèzvù}, \text{all real pros.} \]

otherwise have to do non-archimedean geometry (smoothing doesn't descend); but argue should be the same.