Review:
1. $B_{\infty} -$ nested sequence of $B$-spaces
   - $sc^0, l, k, \infty$ maps preserve levels, and $sc^\infty$ maps whatever it needs to be to make shift maps (smooth reformulations) $sc$-smooth.

2. Certain function spaces can be "compactified" by adding in broken trajectories or nodal curves, via retractors/splicings.

Today:

Def'n: Retractors/splicings.

M-polyfields $\rightarrow$ strong-bundles

Define $q_r = [0, 1] \times \Phi^q_r \times \| \rightarrow [0, 1] \times \Phi^q_r \times \Phi^q_r$.

Define

$\Pi : [0, 1] \times \Phi^q_r \times \Phi^q_r \\
\Pi(v, e, f) = \begin{cases} 
\text{image} \\
(v, e, f) & \text{if } v = 0
\end{cases}$

Claim: $\Pi$ is $sc$-smooth and satisfies $\Pi \circ \Pi = \Pi$. 
\[ k := \pi([0,1] \times \mathbb{R}^{q,b} \times \mathbb{R}^{b,c}) \text{ is our desired} \]

"compactification."

**Defn:** \( r \) is \( \text{s.c.-smooth} \) if satisfies \( r \circ r = r \) \( \text{or} \) \( \text{retract} \) \( k := r(E) \) is our desired "compactification." \( \uparrow \) partial open set in \( r \) (is) not just new \( r \).

Claim: retracts homotopy are the basic building blocks for poly. \( U \) is.

Consider \( \text{Rel. open set in a parameter space} \).

**Defn:** \( V = [0,\infty) \times W \) \( \uparrow \) \( B_{sc} \).

In practice \( \{0,3,7, R^n\} \).

\( \pi : U \times E \rightarrow E \) \( \text{sc.-smooth} \).

\( \pi_0(v,e) = \pi(v) \).

Then \( S = (\pi, E, U) \) is a splicing.

(NB: consider \( r : v,e \rightarrow (v,7i,v,e) \). Then this is a retraction.)

**Defn:** retraction \( R : U \rightarrow U \) \( \text{if} \) \( R(v,e) = (r(e), \pi(e,f)) \) \( \uparrow \) \( \text{linear in f.} \) \( \uparrow \) \( \text{retraction on E} \).

Claim: retracts are building blocks for "new" differential geometry.

How? 1 Topology: \( K^r = r(E) \).

We say \( 0 \leq K^r \text{ is open} \) provided \( r^{-1}(0) \text{ is open in } E \).

2 sc-topology: \( 0 \leq K^r \text{ is k-open} \) provided \( r^{-1}(0) \text{ is open in } E \) \( \text{in topology.} \)
3. Given \( U \subseteq C \subseteq E \) \( \forall u \in U \mapsto Y \) \( v(U) = 0 \).
\( \forall \subseteq Q \subseteq E \) \( \exists : Y \mapsto Y \)

\[ f : \overline{Q} \mapsto \overline{Q} \subseteq K^n \]

Consider the regularity of \( f \) via the following:

\( f \) is \( sc^k \) \( \iff \) for \( v : \mathcal{U} \mapsto F \) is \( sc^k \).

**Defn:**

\[ E / r \quad \overline{Y} \quad \text{top. space} \]

\[ F / w \]

\( \Psi_0 \overline{\varphi}^{-1} \) a scale diffeo. \( \iff \Psi_0 \overline{\varphi} \text{ is } sc^- \).

4. Retracts have tangent bundles,

Suppose \( K \overline{r} = v(E) \overline{r} : 0 \) \( (v : U \mapsto U) \).

\[ v \overline{r} \overline{r} \]

\[ T_r \circ T_r = T_r \] so \( T_r \) is a retraction.

**Defn:** \( T_0 = T_r(T_U) \).

**Note:** This defn is not arbitrary.

\[ y : (-\infty, 3) \mapsto 0 \quad \text{if} \ (v, u) \in T_0 \]

\[ T(y, v) \leq T_0 \].

Then:

\[ y(t) = v(u + tv) \leq 0 \]
\[ f: O \to O', \text{ then} \]
\[ T_{f}: TO \to TO' \]

(Q: Is it easy to see that tangent space doesn't depend on choice of \(r\)?)

A: It's not too hard — we'll see this soon.)

Attempt: \((r, O, \mu, C, E)\)
\[ U \subseteq C \subseteq E \]
\[ r: \text{sc-smooth retraction} \]
\[ r(U) = O, \text{open} \]

Turns out, we don't need \(r\) (or \(U\)?)!

\[ \tilde{r}(\tilde{U}) = O = r(U) \]
\[ \tilde{U}, U \subseteq C \subseteq E \]

Then sc-tp same, and \(T\tilde{r}(T\tilde{U}) = TO = T\tilde{r}(TU)\).

Local model: \((0, 0, E), C \subseteq E\),
\[ 0 \text{ image of some sc retr.} \]

Defn of \(M\)-polyfold: \(\mathcal{U}\) hol charts.

NB: \(\mathcal{U}\) depends on \(E\).

Lies in regards to boundary structure.

Claim: (normally have \(C^{\infty}\) compactness, \(H^{1,p}\) discrete.
\[ \text{need to consider } C^{2} (L^{2} \text{ energy must to get compactness in}) \]