Lefschetz fibrations

\[ \pi : (E^{2n+2}, s) \to (S^1, s_0) \]

Local model for singularities

\[ (\mathbb{Z}_n, \mathbb{Z}_0) \to z_1^2 + \cdots + z_n^2 \]

Assume exact, i.e. \( w = d\theta \).

- Other technical conditions, mostly omitted, but
- The end result is that there is well-defined parallel transport away from \( \text{Crit} \), properly connected given by symplectic complement

\[ \Delta_y = \text{points that parallel transport into the core path.} \]

Local model:

\[ V_y - \text{vanishing cycle.} \]

\[ V_y \text{ Lagrangian in } F_{x} \]

\[ \Delta_y \text{ Lagrangian in } E \]

\[ \Delta_y = U^{5^n} 0\text{-section} \]

\[ D^{n+2} \to \text{Lefschetz thimble.} \]
Think of \( E = (F \times \mathbb{D}^2) \cup n\)-handles via \( V_i \)'s.

Choose a reference point \( p \) paths \( \gamma_1, \gamma_2, \ldots \) to \( \partial \mathbb{D}_i \).

This gives us \( \text{V}_1, \ldots, \text{V}_n \), \( \partial \Delta_1, \ldots, \partial \Delta_n \).

**Fukaya-Donaldson category:**

- **Ob:** \( L \in \mathcal{E} \) Lagrangians
- **Mor:** \( \text{Hom}(L_0, L_1) = HF^*(L_0, L_1) \)
- **Composition:** \( \text{CF}(L_1, L_2) \otimes \text{CF}(L_0, L_1) \rightarrow \text{CF}(L_0, L_2) \)

\[ \begin{array}{c}
L_2 \\
\xymatrix{L_0 & L_1 \ar[r] & E \\
\ar[r] & \sum_n y_n z_n \otimes \langle x \rangle}
\end{array} \]

This gives \& get action or symplectic Floer homology \( E/\text{Ham E} \) on this guy.

**Fukaya category:**

- **Objects:** \( L \in \mathcal{E} \)
- **Morphisms:** \( \text{Hom}(L_0, L_1) = \text{CF}^*(L_0, L_1) \)

\[ \begin{array}{c}
\text{mod:} \quad \text{mod}(Li_{d, 1}, Li_3) \otimes \ldots \otimes \text{mod}(Li_3, Li_2) \rightarrow \text{mod}(Li_0, Li_d)
\end{array} \]

\( n(x_0, x_d, \ldots, x_d) \) structure constants don't mean much by themselves.

\( L \) is defined on different, but together get a unique Floer structure up to quasi-isomorphic.
Then: This defines an $A_n$ category, i.e.

$$
\sum_{m,n} \mu^{d-m+1} \left( x_d, x_{n+m} \mu^{m} x_{n+m}, x_{n+m}, \ldots, x_0 \right) = 0.
$$

(Pf: $0 = \varnothing \left( \bigcup_{p_k} \bigcup_{m,n} x_0 \ldots x_{n+m} \right)$)

$$
\sum_{n \neq m} \mu^{n-m+1} x_0 \ldots x_{n+m}
$$

Dealing with the self-intersection problem:

Simplification: Directed Fukaya's category

Obj: $L_1, \ldots, L_n$

Mor: $\hom(L_i, L_j) = \begin{cases} 
\hom(L_i, L_j) & \text{if } i \geq j \\
eq i, k & \text{if } i = j \quad (\text{in general,} \\
0 & \text{else.} \\
H^*(L_i^2)
\end{cases}$

$\mu^d: \hom(L_{i_0}, L_{i_d}) \otimes \cdots \otimes \hom(L_{i_{d-1}}, L_{i_d}) \rightarrow \hom(L_{i_0}, L_{i_d})$ is normal if $i_0 < \cdots < i_d$,

$0$ if $d \geq 3$ and not $1$, and

$\varnothing$ acts as identity in $\mu^2$. 

Intuition:

\[ \Delta_1 \xrightarrow{\sim} \Delta_2 \]

Define HF for this:

\[ \text{move left 90° ccw., } \phi \]

So \( \text{Hom}(\Delta_1, \Delta_2) = \text{Hom}(\phi(\Delta_1), \phi^{-1}(\Delta_2)) \)

So \( \text{hom}(\Delta_2, \Delta_1) = 0 \) by \( \phi(\Delta_2) \cap \Delta_1 = \emptyset \)

and

\[ \text{Hom}(\Delta_1, \Delta_1) \text{ looks like } \]

\[ \cup \]

One distinct generat.

Related to

Legendrian contact homology algebra
Legendrian quantum homology?

What if \( L_1, L_2 \) not transverse?

pick hamilt. perturbations for each pair \( L_i, L_j \):

\[ L_2 \xrightarrow{\text{hamilt. isotopy}} L_2' \]

\[ L_1 \xrightarrow{\text{hamilt. isotopy}} L_1' \]

Depict hamilt.

Big complicated induction to give you coherent perturbations.

This well defined up to \( \text{Ass quasi iso.} \)

--- Break
**Categories**

**A**

- **Directed**
  - $\text{Obj. } \mathcal{L}$
  - $\text{hom}(L_i, L_j) = \begin{cases} CP^k(V_i, V_j) & \text{if } i < j \\ 0 & \text{else} \end{cases}$

**B**

- **Undirected**
  - $\text{hom}(L_i, L_j) = (CP^*(V_i', V_j'))$

These depend on paths. How?

![Diagram of paths and transformations]

Thus, that any

$\mathcal{V}_i = \mathcal{T}_{V_i} V_{i+1}$

Every other collection of paths is related by a sequence of such moves.

**Tw A** - triangulated hull

**Obj** "complexes or objects in $\mathcal{A}$"

- $\{A_i, \beta_i\}$ ordered collection of $A_i \in \text{Ob } \mathcal{A}$
  - $\Delta_{ij}$ upper/lower triangle
  - $\sum_{i < \cdots < k} m_{ik}(A_{i+1}, \cdots, A_k) = 0$
In $\text{Tw} A$, $X, Y \in \text{Ob} \text{Tw} A$.

$L_X Y = \{ \otimes \text{Hom}(X \otimes i[j], Y) \otimes X(-i[j]) \to Y \}^\alpha$

\[ x \xrightarrow{\alpha_1} \]
\[ x \xrightarrow{\alpha_2} Y \]
\[ x \xrightarrow{\alpha_3} \]

In $\text{Tw} A$,

$Zv W = L v W$. [Serre's exact sequence for Dold twist]

$V_1 \to V_\lambda \to V_\lambda, \ldots$

\[ \downarrow \text{within} \]

$V_1 \to V_\lambda \to V_\lambda, V_\lambda, V_\lambda \to \ldots$

This translates changes at the same.