A unital \(k\)-algebra

\[ M \] \(A\)-bimodule

**Hochschild chain complex**

\[ C_n(A, M) := M \otimes_k A^n \]

\[ d_i : (m \otimes a_1 \otimes \ldots \otimes a_n) = \begin{cases} (m \cdot a_1) \otimes a_2 \otimes \ldots \otimes a_n & i = 0 \\ m \otimes a_1 \otimes \ldots \otimes (a_{i-1} \cdot a_i) \otimes \ldots \otimes a_n & i = n \\ (a_n \cdot m) \otimes a_1 \otimes \ldots \otimes a_{n-1} & i = n \end{cases} \]

**Note this map!**

\[ s_i : C_n \to C_{n+1} \]

\[ s_i : (m \otimes a_1 \otimes \ldots \otimes a_n) := (m \otimes a_1 \otimes \ldots \otimes a_{n-1} \otimes 1) \]

\[ \text{HH}_*(A, M) := H_*((C_*(A, M), d), \text{on} C_n, d = \sum_{i=0}^n (-1)^i d_i) \]

\[ d' = \sum_{i=0}^{n-1} (-1)^i d_i \]

\((C_*(A, M), d)\) is called the cyclic bar complex

\[ B^c(A, M) \]

\[ \big( n = 0 \big) \quad C_0(A, M) = M \]

\[ C_1(A, M) = M \otimes A \]

\[ \text{HH}_0(A, M) = M / \{ ma-am | m \in M, a \in A \} \]

If \(M = A\), \(A\) commutative, \( \text{HH}_0(A, A) = A \)
\[ n = 1 \quad M = A, \quad A \text{ commutative} \]

\[ HH_1(A) = A \otimes A / \{ a_0 a_1 a_2 - a_0 a_2 a_1 + a_2 a a_1 \} \]

\[ \cong \Omega^1(A) \text{ Kähler differentials} \]

\[ \text{gen by symbols } da \text{ and so that } d(ab) = da \cdot b + b \cdot da \text{ etc.} \]

Remark. In general for smooth algebras, the Hochschild-Kostant-Rosenberg theorem states that \[ HH_\ast \cong \Omega^\ast(A) \]

**Hochschild cochain complex**

\[ \delta \text{ Hochschild } \quad C^n(A, M) = \text{Hom}_k(A^\otimes n, M) \]

\[ HH^2 = \text{deformations} \quad HH^3 = \text{obstructions} \]

@ Cup product \( f \in C^n(A, A), g \in C^m(A, A) \)

\[ f \circ g \in C^{n+m}(A, A) \]

This descends

@ compositions \( a : C^n \times C^m \to C^{n+m-1} \)

\[ f \circ g (a_1, \ldots, a_{n+m-1}) = f(a_1, \ldots, a_n, g(a_{n+1}, a_{n+2}, \ldots, a_{n+m-1})) \]

\[ \phi(n) = \text{Hom}_k(A^\otimes n, A) \]

\[ \psi(n) \]
One combines \( f \circ i \cdot g \)

\[
fog = \sum (-1)^{(n-1)(i-1)} f \circ i \cdot g
\]

Finally define

\[
[f, g] := fog - (-1)^{i+1} gof
\]

This is a Lie bracket of degree \(-1\) that descends to homology.

Composites satisfy "obvious" associativity rules.

Moreover, we have \([a, b, c] := [a, [b, c]] + (-1)^{i} [b, [a, c]]\)

In terms of Gerstenhaber

\(\text{HH}^*(A), [\cdot, \cdot], \mu\) is a Gerstenhaber algebra.

Precisely, structure of \(\text{HH}^*(A), [\cdot, \cdot]\) present in Hochschild cohomology.
Part II

\( V := \{ V_i \} \) graded vector space

\( \mathcal{V} := \{ V_i = \text{Hom} (V_i; k) \} \) graded dual.

There is an isomorphism of complexes

\[
C_\otimes^\mathcal{V} (A, A) \cong C^\mathcal{V} (A, A^\mathcal{V})
\]

\( \varphi : A^\otimes^{n+1} \to k \quad \psi : A^\otimes^n \to \text{Hom}(A, k) \)

\( \text{HH}_\otimes^\mathcal{V} (A) \cong \text{HH}^\mathcal{V} (A, A^\mathcal{V}) \)

Cyclic Homology

\( C_n (A, A) = A \otimes A^\otimes^n \)

There is an action of \( \mathbb{Z}_{n+1} \) on \( C_n (A, A) \)

given by \( t (a_0 \otimes \ldots \otimes a_n) = (a_n \otimes a_0 \ldots \otimes a_{n-1}) \)

Lemma \( (1-t)d' = d(1-t) \)

\( d'N = Nd' \)  where \( N = 1 + t + \ldots + t^n \)
The Cyclic Bicomplex \( CC(A) \)

\[
\begin{array}{cccccc}
A & \xleftarrow{1-t} & A & \xleftarrow{1-t} & A & \xleftarrow{1-t} & A \\
A & \xleftarrow{1-t} & A & \xleftarrow{1-t} & A & \xleftarrow{1-t} & A \\
A & \xleftarrow{1-t} & A & \xleftarrow{1-t} & A & \xleftarrow{1-t} & A \\
\end{array}
\]

By def. \( HC_\ast(A) = H_\ast(\text{Tot} \; CC(A)) \)

Thm (Cooms' periodic exact sequence) \( \text{Connes' } B \) operator.

\[
\text{long exact sequence}
\]

\[
\begin{array}{cccccc}
\rightarrow HH_n(A) & \rightarrow HC_n(A) & \rightarrow HC_{n-2}(A) & \rightarrow HH_{n-1}(A) \\
\end{array}
\]

\[
\begin{array}{cccccc}
\rightarrow CC^{\{2,2\}}(A) & \rightarrow CC(A) & \rightarrow CC(A)[2,0] & \rightarrow 0 \\
\end{array}
\]

Just a note: Reminds of \( S^1 \)-equivariant homology (Gysin sequence).

Cyclic Homology \( \Rightarrow S^1 \)-equivariant homology. 
Relation with the free loop space

Good willie, $M$ connected

$HH^*_X(C_*(S^1M), C_*(S^1M)) \cong H^*_X(LM)$

$S^1M =$ based Moore loops. $LM = $ free loop space.

Extension to dga $5$. $A$ dga over $K$

$M$ dga mod one $A$

$C_n(A, M) = (S^1M) \otimes (S^1A)^{\otimes n}$

$(S^1V)_i = V_{i+1}$

$d : C_n \to C_{n-1}$

$d = \bar{d}_1 + \bar{d}_2$

$\bar{d}_1$: old $d$ with signs

$\bar{d}_2$: tensor product $d$

$a \in (S^1M) \otimes (S^1A)^{\otimes n}$

multiplieaken on reduced cubical chains

Since $M$ simply connected

$HH^*_X(C^*M, C^*M) \cong H^*_X(LM)$

Identity

$HH^*_X(C^*M, C_*(M)) \cong H^*_0(LM)$

Can use Poincaré here, tricky.
Thm (Adams)

\[ M \text{ simply-connected} \]

\( (C_*(M), \partial) \) counted, coassociative, coaugmented by cochain \[ A : C_1M \to C_0M \otimes C_0M \]

split into back and front trees \[ \Sigma(C_*(M)) = \text{as above construct on } (C_*(M), \partial) \]

\[ (T_s(C_*(M)), d = \partial + A) \]

is homotopy equivalent to chains on loop space \[ C_*(\Omega M) \text{ as dga}. \]

**Picture behind Goodwillie's Thm**

G top group \( X = G \) 2-sided G-space

\[ B^\text{cycle}(G, G) = \bigcup_{k \geq 0} G \times G^k \]

\( d_i, s_i \) as before make \( B^\text{cycle}(G, G) \) into a cyclic space.

\[ B . G = B^\text{cycle}(G, \{x\}) \]. Clearly there is a projective map

\[ p : B^\text{cycle}(G, G) \to B . G \]

\[ (g_0, g_1, \ldots, g_n) \to (x, g_1, \ldots, g_n) \]
Claim 2: The geometric realization of a cyclic space carries an action of $S^1$.

$$S^1 \times |B^c(G,G)| \rightarrow |B^c(G,G)| \rightarrow |B(G)| = BG$$

Claim 3: The adjoint map

$$|B^c(G,G)| \rightarrow \text{Map}(CS^1BG)$$

is a homotopy equivalence.

Claim 4: We can replace $G$ by $SM$ in this construction.

$$|B^c(SM,SM)| \cong L(BSM) \cong LM$$

Pass to chains.

Also

$$HC_\ast(C_\ast(BSM), C_\ast(SM)) \cong H_\ast^{S^1}(LM)$$