Example 1. \( \Delta \subseteq M_R \) a lattice polytope (compact)

\[ M = \mathbb{Z}^n, \quad M_R = M \otimes \mathbb{R} \]

\( m \rightarrow \) gives a projective toric variety \( P_\Delta \).

\[ C(\Delta) \subseteq M_R \otimes \mathbb{R} \]

\[ \{ (rm, r) \mid m \in \Delta, r > 0 \} \]

\( C(\Delta) \cap (M \otimes \mathbb{Z}) \) is a finitely generated monoid.

\[ P_\Delta = \text{Proj} \ k [ C(\Delta) \cap (M \otimes \mathbb{Z}) ] \]

with \( \text{deg} \ z^{(m, d)} = d \)

This gives an ample line bundle \( L = \mathcal{O}_\Delta(1) \) on \( P_\Delta \)

\[ \Gamma (P_\Delta, L^{\otimes d}) = k [ C(\Delta) \cap (M \otimes \mathbb{Z}) ]_d \]

This has a canonical basis indexed by points \( (m, d) \in C(\Delta) \cap (M \times \mathbb{N}) \)

\[ \text{i.e. } m \in (d \Delta) \cap M. \]

\[ \text{i.e. } m \in \Delta \cap (dM). \]
Example 2: \( \Delta \subseteq \mathbb{M}_{\mathbb{R}} \) a lattice polytope

Let \( \mathcal{P} \) be a polyhedral decomposition of \( \Delta \) into lattice polytopes.

\[ \Psi: \Delta \rightarrow \mathbb{R} \] PL w.r.t. decomposition, and

\[ \Psi \] should have integral slopes.

\[ \Delta_\Psi \subseteq \mathbb{M}_{\mathbb{R}} \oplus \mathbb{R} \text{ polyhedron} \]

\[ \{ (m, r) \mid m \in \Delta, r \geq \Psi(m) \} \] e.g.

modified cone:

\[ \mathcal{C}(\Delta_\Psi) = \{ (dm, dr, d) \mid (m, r) \in \Delta_\Psi, d > 0 \} \]

\[ \psi \text{ closure adds } \{(0, r, 0) \mid r > 0\} \]

\[ \mathcal{P}_{\Delta_\Psi} = \text{Proj } k \left[ \mathcal{C}(\Delta_\Psi) \cap (\mathbb{M} \oplus \mathbb{Z} \oplus \mathbb{Z}) \right] \]

The degree zero part of this ring is \( k[\mathcal{E}] \), where \( \mathcal{E} = \mathbb{Z}^{(0,1,0)} \),

giving a projective morphism (as this ring is a \( k[\mathcal{E}] \) module).

\[ \pi: \mathcal{P}_{\Delta_\Psi} \rightarrow \text{Spec } k[\mathcal{E}] = \mathbb{A}^1 \]

\[ \mathcal{U}_1 \]

\[ \mathcal{P}_{\Delta_\Psi} \times \mathbb{A}^2 \backslash \{0\} \]

\[ \text{two variety given by } \sigma \]

\[ \pi^{-1}(0) = \bigcup_{\sigma \in \mathcal{P}_{\Delta} \cap \mathbb{M}} \mathcal{P}_{\Delta} \]

We have a relatively ample line bundle \( L \),

canonical basis for \( \Gamma(\mathcal{P}_{\Delta_\Psi}, L^{\otimes \delta}) \) indexed by \( \Delta_\Psi \cap \mathbb{F}_\delta (\mathbb{M} \oplus \mathbb{Z}) \)

If we take \( m \in \Delta(\frac{1}{d} \mathbb{Z}) \), then \( (m, \Psi(m)) \in \Delta_\Psi(\frac{1}{d} \mathbb{Z}) \)

\[ \Delta^{\frac{1}{d} \mathbb{M}} \]

\[ \Delta_\Psi \cap \frac{1}{d} (\mathbb{M} \oplus \mathbb{Z}) \]
defines a section of $L$ which, when restricted to $\pi^{-1}(0)$, is the type of section considered as before, extended by 0.

Ex.

$
\begin{array}{|c|c|}
\hline
\pi \times \Psi' & \Psi' \\
\hline
\hline
\end{array}$

(gives section of 1.b. on $\pi \times \Psi'$ that
\begin{itemize}
\item gives section of 1.b. on $\pi \times \Psi'$ that
\item glue along boundary, 0 elsewhere.
\end{itemize}

We also have a canonical lifting, the family.

Example 3: $\Gamma \leq M$ a sublattice

$\varphi$ a lattice polytopal decomposition of $M_{\mathbb{R}}$, which is $\Gamma$-periodic.

$\varphi: M_{\mathbb{R}} \rightarrow \mathbb{R}$ PL: w.r.t. $\varphi$, strictly convex, internal slope, and.

for $\gamma \in \varphi$, $\varphi(\gamma) = \varphi(\gamma) + \gamma(\gamma)$ with $\gamma$ affine linear.

(e.g. $M = \mathbb{Z}, \Gamma = d\mathbb{Z}$)

Tropical theta function.

$\varphi(\gamma) = \left\{ \sum_{i \in \mathbb{Z}} -i(i+1) \right\} \gamma^{\gamma}$

$\varphi = \max \{ i \gamma - \gamma \left( i \gamma \right) | \gamma \in \mathbb{Z} \}$

For example,:

$P_{\Delta} = \text{Proj } k[\mathbb{C}(\Delta) \cap (M \times \mathbb{Z} \times \mathbb{Z})]$

General fiber is $M^v \otimes G_m$ (algebraic torus)

(so hardly proper)
Special fiber is an infinite chain of $\mathbb{P}^1$'s:

For example, how to deal with this?

$\Gamma$ acts naturally on $\mathbb{P}_{\Delta_0}$. (In a special fiber, shifts $\mathbb{P}^1$'s up by $d$.)

One way to see this:

Normal fan:

---

Try to divide by $\Gamma$, but this is a "hellish" thing to do.

- One way of dealing with this: Complex analytic case:

\[
\pi^{-1}(D) \subseteq \mathbb{P}_{\Delta_0} \\
\downarrow \quad \downarrow_{\pi} \\
D \subseteq \mathbb{A}^1
\]

$\pi^{-1}(D)/\Gamma$ makes sense.

One-dim $\mathbb{C}$ example:

Fibre over $t \in \mathbb{A}^1$ is $\mathbb{C}^* / \mathbb{Z}$. If $t = 1$, we run into trouble, but in $D$, action is properly discontinuous.

- Let's do this formally:

\[
\begin{array}{c}
\begin{array}{c}
\subset \mathbb{C}(\Delta_0) \\
\ni
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\gamma_1 \mapsto T_{\gamma} \in \text{Aut}(M \oplus \mathbb{Z} \oplus \mathbb{Z})
\end{array}
\]

with properties

1. $T_{\gamma}(\mathbb{C}(\Delta_0)) = \mathbb{C}(\Delta_0)$

2. $T_{\gamma}$ preserves degrees (acts trivially on last component).

3. $T_{\gamma} |_{\mathbb{M} \oplus \mathbb{Z} \oplus \mathbb{Z}} = \begin{pmatrix} \text{Id} & 0 \\ 0 & 1 \end{pmatrix} \circ [\text{This uniquely determines } T_{\gamma}]$.

Sections of line bundle of quotient are just equivalent sections here.
... (construct equivalent sections by taking $\mathcal{E} X_\sigma(a)$ for any sect $a$ ...)

c.g. $T_d = \begin{pmatrix} 1 & 0 & d \\
                        d & 1 & d(d-1)/2 \\
                        0 & 0 & 1 \end{pmatrix}$

Given $m \in \frac{1}{\delta} \mathbb{Z}$, define $\sum_{\gamma \in \Gamma} T \sigma(\beta, \gamma v(m), d)$ (intrinsic sum but can check well-defined, otherwise # of terms at any t).

$\uparrow$

(in analytic category, these would converge, $\mathcal{E}_\sigma$ $\Theta$-functions.)

This gives a well-defined section of $\mathbb{X}^{\Theta_{x,y}}$ on $\text{IP}_{\Delta_y}^\Gamma$ for every $m \in \frac{1}{\delta} \mathbb{Z}$, $\mod \Gamma$.

Let $B = \mathbb{M}_{\mathbb{R}}/\Gamma$

$me B(\frac{1}{\gamma} \mathbb{Z})$

[Q (Krummle): Now that we have this IP embedding, can we do tropical elimination at this point (cf. Tevelev)? This would tell us...]

A: Probably. Interesting question.

With the action of $\Gamma$, $\mathbb{M} \otimes \mathbb{Z} \otimes \mathbb{Z}$ descends to a local system $\mathcal{M}$ on $B$.

\[
\begin{array}{c}
0 \rightarrow \mathbb{Z} \rightarrow \text{Aff}(\mathcal{M}, \mathbb{Z}) \rightarrow \Lambda^* \rightarrow 0 \\
0 \rightarrow \mathbb{Z} \rightarrow \mathcal{M} \rightarrow \text{Aff}(\mathcal{M}, \mathbb{Z})^* \rightarrow 0
\end{array}
\]

$\Lambda$ = integral

vector fields  on $B$.

$\mathcal{M}$  on $\mathbb{R}(\text{structure})$.

$\mathbb{Z}$, quotient sheaf, degree.

$\mathbb{Z}$  on $B$.

(can see from $T_d$ to $b$, it preserves the $\otimes\mathbb{Z}$, degree, etc.)

$\otimes$ of $B$

$B =$ Legendre

transform

subshaf

comes from $\mathcal{L}(r)$ structure.

subshaf, quotient sheaf

(closed under degree, etc.)

$\otimes$ of $B$
$\Theta_{m,s} = \sum_{\text{all straight lines joining } m \text{ and } x} \text{parallel transport of some } \tau_{m,x} \text{ to } x \text{ via the line } \ell$

some stalk at $m$.

Canonical element.

$\ell \in \mathcal{B} \left( \frac{i}{s}, \mathbb{Z} \right)$

Adding singularities to the affine structure (Philosophy: allow lines to "bend").

Simple example: polyhedral decomposition of plane into upper Blower, singularity at $0$.

Corresponds to $\mathbb{C}^* \times \mathbb{C}^2$

$\text{corresponds to}$

Could just try to draw a straight line & parallel transport the monomial.

but it's discontinuous through singularity

point is this is no longer a straight line.

Need to sum over broken lines:
as you move across the singulars, straight might change to broken & vice versa.

Everything can be described as a sum of broken things, to make sure things are well-defined.

Q: (Above aid) Why call these broken lines & not balanced trees? (they're not always the same.)

This is currently formal only.

Interesting question: Is there a characterization of the sections we’re building so that they can be pushed forward analytically?