$R$ - DVR
$k, K, K = k[[t^i, z^j]]$

Implicitization:
$\alpha: Y \longrightarrow G^n_m, K, X = \alpha(Y)$ sch. theoretic image.


(this tells us the Newton polytope/monomials at least for hypersurfaces but not the coeffs).

1. compute trop $X$ from geometry of $Y$.
2. $\longrightarrow$ trop $Y$.
3. coarsest polyhedral structures on trop $X$ and its topology.

Bieri - Groves definition of trop $X$:

valuation: $\langle \text{val} \rangle$

ring valuation: $K[\mathcal{X}] \longrightarrow L = \mathbb{Q}[z, \ldots, z_n]$ valuation $\psi$

Let $z_1, \ldots, z_n$ be coordinates on $G^n_m, K$.
$[\text{val}] = [\text{val}(z_1^i), \ldots, \text{val}(z_n^i), \text{val}(t)]$

Def: trop $X = \{ [\text{val}] | \text{val} \text{ runs over all ring valuations } \}$

Thm: (Kapranov) $\forall x \in X(\mathbb{R})$

$\text{val}_x: K[\mathcal{X}] \longrightarrow \mathbb{R}_+$

$\text{val}_x: f \longrightarrow f(x)$

$trop(X) = \{ [\text{val}_x] | x \in X(\mathbb{R}), v \in \mathbb{R}_+ \}$
Field of lins:

\[
\begin{array}{c}
\text{Thm: } K[X] \xrightarrow{\text{val}_D} \Omega_{\text{div}} \\
D \text{ is a divisor on a normal } R\text{-scheme birational to } X. \text{ Then}
\end{array}
\]

\[
\text{typ } (X) = \{r \cdot [\text{val}_D] : D \text{ is a divisor, } r \in \mathbb{Q}\}
\]

\[
\text{Thm: (Gubler, Hacking-Leclerc-T)}
\]

In the implicitization setup, \( Y \to X \to G_m^m \), suppose \( Y \) admits a proper, semi-stable model \( \overline{Y} \). (Resolution of \( Y \) with normal crossing boundary).

Then \( \overline{Y} \Delta = \overline{Y \setminus Y} = D_1 \cup \cdots \cup D_r \).

Consider a simplicial complex such that \( \{ D_{i_1} \cap \cdots \cap D_{i_k} \} = \emptyset \) is a simplex \( \iff D_{i_1} \cap \cdots \cap D_{i_k} \neq \emptyset \).

Then \( \text{typ } X = \text{union of all these cones } F(\sigma) \).

**Example:** \( \mathbb{A}^2 \)

\[
\begin{align*}
\text{Let } & P^2 \\
& x^2 + y^2 + z^2 = 0 \\
& x^2 + y^2 = 0 \\
& x^2 = y^2 \\
& x^2 = y^2 = 0 \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{Y} & = X \\
\overline{\mathcal{Y}} & = \mathbb{P} \mathcal{B}_0 P^2 \\
\end{align*}
\]

\[
\begin{align*}
(x) & = L_1 + E - L_0 \\
(y) & = L_2 + E - L_0 \\
(x+y-1) & = L_4 - L_0 \\
\end{align*}
\]

(transpose this matrix to write down vectors)
\[ [L_3] = l_3 \]
\[ [L_2] = l_2 \]
\[ [L_1] \text{ (as along } l_1, \text{ only } x \text{ vanishes) } \]
\[ [L_4] = l_4 \]

Intersect with sphere \( S_3 \):

\( S_3 \): this is the link of trap \( X \).

\[ \begin{align*}
S_3 & : \\
& \\text{this is the link of trap } X.
\end{align*} \]

Ex. 2: \( \overline{1/3} \)

\[ \begin{array}{c}
\text{Spec } R.
\end{array} \]

\[ \begin{array}{c}
\text{object blown up}
\text{on surface, now}
\text{on a three-fold,}
\text{i.e. a family}
\end{array} \]
(2) compute the $X$ from $\text{trop } Y$:

Suppose:

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{dom}} & X \\
\downarrow & & \downarrow \\
\mathbb{G}_m^k & \xrightarrow{\text{homo}} & \mathbb{G}_m^n
\end{array}
\]

Then,

\[
\begin{array}{ccc}
\text{trop } Y & \xrightarrow{\text{trop }} & \text{trop } X \\
\downarrow & & \downarrow \\
\mathbb{Q}_m^k & \xrightarrow{\text{top}(x)} & \mathbb{Q}_m^n
\end{array}
\]

(i.e., $dx$)

Remark: For any $X$, can take

\[\mathcal{O}^*(X)/\mathbb{R}^* \simeq \mathbb{Z}^r \text{ (Samuel)}\]

Any $X$ has a canonical $\xrightarrow{\xrightarrow{\text{G}_m/k}}$ intrinsic torus of $X$.

If $X$ embeds in the torus, it also embeds in the intrinsic torus.

(best possible embeddings)

\[\xrightarrow{\text{subvarieties in tori}} \xrightarrow{\text{subvarieties in Ab.varieties}} \xrightarrow{\text{subvarieties in } \mathbb{P}^n} \]

\text{(Albanese map). \ \ \subvarieties in } \mathbb{P}^n.
If \( Y \to X \) any morphism
\[ Y \to G_m, \]
then I open \( Y_0 \subset Y \) very affine, closed subspace of an alg. torus, so
\[ Y_0 \to X \]
\[ U \]
\[ Y_0 \text{ very affine.} \]
\[ G_m \]
\[ G_m \quad \text{homo.} \]
so (always by a monomial map).

\[ \text{(intrinsic torus)} \]

\[ \text{Fix:} \]
\[ M_{0,n} \]
\[ \text{forget } \mathcal{P}_n \]
\[ M_{0,n-1} \]

\[ M_{0,n} \subset \mathcal{G}_m^{n(n-3)/2} \]
\[ \text{homo.} \]
\[ M_{0,n-2} \to \mathcal{G}_m^{(n-1)(n-4)/2} \]

so get \( \text{trop}(M_{0,n}) \subset \text{fan of dim } n-3 \)

fibers are exactly
phylogenetic trees, i.e.
map, rat'l curves. If
you restrict them by length of
interior components, get exactly \( M_{0,n} \).
There is a refined version of the above theorem which discuses multiplicities.

**Thm:** (Sturmfels - T) version w/ mults.

Check using intersection theory that the right things happen.

**Ex:**

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & -1 & -1 & 0
\end{pmatrix}
\]

\[f : G_m^3 \to G_m^6\] defined by 6 Laurent polynomials

\[w/\text{Newton polytopes } [m_{0,i}, 2] \times [m_{1,i}, 2] \times [m_{2,i}, 2], \quad i = 1, \ldots, 6\]

with generic coeffs.

\[T = \text{trop } f(G_m^3)\]

\[\text{T does not admit a coarsest fan structure (because its smooth locus is non-convex).}\]
Link $T \subset S^5$ has a non-trivial 1st homology group. Link $\text{trp}(M_{(n)}) = \text{a bouquet } (n-2)! \cdot (n-3)$-dim. spheres \[\text{has only top reduced homology.}\]