Today: mirror symmetry for toric varieties

Recall: complex geometry $\leftrightarrow$ symplectic geometry

complex subvar. \{ holom. bundles \} $\leftrightarrow$ Lagrangian submanifolds

Toric Varieties:

(partial) compactification of $(\mathbb{C}^*)^n$ s.t. torus action extends.

Can describe by \{ complete fan \} (complete) fan

e.g. $\mathbb{C}P^2$

affine charts = maximal cones

$\psi_1 \rightarrow \psi_i$ gluing given by combinatorics.

If we equip w/ a polarization (= projective embedding), then we can describe it by a lattice polytope $\Delta$ (moment polytope), dual to the fan.

Facets $<\psi_i, x> = \alpha_i$, $\psi_i \in \mathbb{Z}^n$ rays of fan, $\alpha_i \in \mathbb{Z}$.

vertices $\leftrightarrow$ max. cones

Ex. $\mathbb{C}P^2$

* $\Delta$ = orbit space for $T^n$-action on $X$

(the polarization gives us a symplectic form induced by the proj. space in which we embed.)
Symplectic of $\mathbb{CP}^{k}$ at edges $\iff$ degrees of $\mathbb{CP}^{1}$'s inside ambient $\mathbb{CP}^{n}$ $\iff$ lengths of edges w.r.t. integer lattice.

$\mathbb{CP}^{2} \rightarrow \mathbb{CP}^{4}$

all degree 3 homog. monomial

$(x:y:z) \mapsto (x^3:x^2y:xyz)$

The mirror [Batyrev, Givental, ...]

$X_{\Delta}$ toric var. with facets $<v_i, x> = \alpha_i$

$\Rightarrow$ the mirror is $(\mathbb{C}^{*})^n$ equipped with Laurent poly $W(z) = \sum_{\text{facets}} t^{\alpha_i} z^{v_i}$

Ex. $\mathbb{CP}^{2}$

\[ (-1,0) \quad x_2 = 0 \quad (1,0) \]

\[ -x_1 - x_2 = 1 \quad x_1 = 0 \quad (1,0) \]

\[ (0,-1) \]

to rays of fan (vertex normal)

$\iff (\mathbb{C}^{*})^2$, $W = z_1 + z_2 + \frac{t}{z_1 z_2}$

$\xleftarrow{\text{superpotential}}$

$t$ parameter, $t \rightarrow 0 \equiv \text{Large ex. structure limit}$.

What does mirror symmetry predict about $X_{\Delta}$ vs. $W$?

Various things:

- $QH^{*}(X_{\Delta}) \cong \text{Jac}(W) := \mathbb{C}[[t]][z_i^{\pm 1}]$

quantum cohom. ring Jacobian ring $\langle z_i, W \rangle$
In our case, both equal \( c_k \big/ h^3 = 6.1 \)

* and coherent sheaves on \( X_\Delta \) (complex subvarieties, vector bundles, \( \ldots \))

\( \leftrightarrow \) Lagrangian submanifolds in \( (\mathbb{C}^*)^n \) with boundary \( \partial L \subseteq W^{-1}(-1) \) (non-constant)

\( \text{(ex. geometry of } X_\Delta \text{) } \leftrightarrow \text{ symplectic geometry of } (\mathbb{C}^*)^n \text{ with } \exists W = 13 \)

\[ \{ W = 13 \text{ is (after some modification) mirror to a Calabi-Yau hypersurface in } X_\Delta \} \]

\[ H_0 = \mathcal{U} \text{ (toric strata) } \]

[Mohammed Abouzaid's thesis]

Viewing \( \{ W = 1 \} \via \text{ tropical geometry } (t \to 0) \)

\( \log_{1/6} (W^{-1}(-1)) : \) amoeba converges to

\( \text{ tropical hypersurface } \mathbb{T} \subseteq \mathbb{R}^n \)

(= locus where \( \mathfrak{R} \) monomials in \( W \) are tied for largest)

\[ \text{Ex. } \mathbb{C}P^2 : W = 1 = z_1 + z_2 + \frac{t}{z_1 z_2} - 1 \]

"Tropicalize" \( \max (x_1, x_2, 1 - x_1 - x_2, 0) \)

[Diagram]

\[ 1 - x_1 - x_2 \]

general fact: one component of \( \mathbb{R}^n - \mathbb{T} \) is \( \Delta \)

(If non-Fano, we'd have more than one bounded amoeba).
We want: Lagrangian submanifolds in \((\mathbb{C}^*)^n\) whose image by \(\log_{e^k}\) is \(\Delta\).

and boundary \(=\) complexification of faces of \(\Delta\).

(no shift in coamoeba direction \(\Delta\) picked constant to be real pos./negate)

i.e. in coamoeba, edge \(k\) looks like \(\bigotimes\).

We'll construct mirrors to the line bundles \(O(k)\) over \(X_\Delta\).

Holom. line bundle = where defining sections of alg. subs. live.
Ex: \(\mathbb{CP}^2\), \(\Sigma\chi^3 + \gamma^3 + \zeta^3 = 0\) (not a function.

over an affine subset, can view it as a function \((\frac{x}{\zeta}, \frac{\gamma}{\zeta}, 1)\)

\(x^3 + \gamma^3 + \zeta^2\) is a section of \(O(3)\)

= sheaf of regular functions which scale homogeneously with weight 3.

Given a projective variety, \(O(k)\) = bundle containing deg. \(k\) homog. polynomials

- Want: Lagrangian \(L_k \subset (\mathbb{C}^*)^n, W^{-1}(1))\) mirror to \(O(k)\) over \(X_\Delta\)?

- \(L_k\) should be a graph \(\Sigma f = f(x)^3\)

\((x, p) \in \mathbb{R}^n \times (\mathbb{S}^1)^n, \chi \in \Delta\)

\(x, p \quad f: \Delta (c \mathbb{R}^n \setminus 0) \to T^n\).
Justification: the bundle has
\[ L_k := \left\{ (x, p) \in \Delta \times (\mathbb{R}/\mathbb{Z})^n \mid p_i = -k x_i \right\} \]
\[ \text{(this involves choosing a metric, i.e. identify } \mathfrak{T}X = T^*X) \]

Check: 
- \( L_k \) is Lagrangian for \( \Sigma \) \( dx_i \wedge dp_i \)
- \( A^+ \exists L_k : \text{when } x \in F \text{ facet of } \Delta \)

\[ \langle y, x \rangle = \alpha \in \mathbb{Z} \]

\[ F^\alpha = \left\{ (x, p) \mid \langle y, x \rangle = \alpha \right\} \]

\[ \text{Now: points of } L_0 \cap L_k = \left\{ x \in \Delta \mid -kx \equiv 0 \text{ mod } \mathbb{Z} \right\} \]

\[ \subseteq \mathcal{F}_\Delta \cap \left( \frac{1}{k} \mathbb{Z} \right)^n \]

= \text{integer points in } k \Delta.

Cf. Classical fact:
- \( F \) basis of global sections of \( O(k) \) given by integer points in \( k \Delta \).
- \( \mathbb{C}P^2, O(3) \)\[ x^3 \]

\[ x^2 y \]

\[ x y^2 \]

\[ y^3 \]

[Diagram of \( \mathbb{C}P^2, O(3) \) with integer points labeled and a triangle.]
Hence: \( \text{Hom}(\mathcal{O}, \mathcal{O}(k)) \cong \text{H}^0(L_0, L_k) \)
\( (\text{for } k > 0) \cong \mathbb{C}^{k+1} \cap \mathbb{Z}^n \)

Ex: \( CP^1 \) sections of \( \mathcal{O}(1) = q_0 x + q_1 y \)
\[ 1 \to \mathcal{O}(2) = q_0 x^2 + q_1 xy + q_2 y^2 \]
\[ 1 \to \]

\((\mathbb{C}^*), \ W = z + \frac{\ell}{z} \rightarrow \)

\( W^{-1}(1) = 2 \) points

Check: \( x, y = xy \)?

\[ \mathcal{O} \xrightarrow{\times} \mathcal{O}(1) \xrightarrow{\times} \mathcal{O}(2) \]

\[ xy \]

Counterpart:

\[ L_0 \xrightarrow{\times} L_1 \xrightarrow{\times} L_2 \]

How to compute this?

Ans: count above such triangles, sum over all \( \ell \) w/ coeffs the number of such triangles.

Claim: in this case, there is only one such triangle. See above figure (†) for the triangle and what "\( xy \)" should be. Matches with complex case!