Last time: In Alg, general theorem that you can flatten any morphism by blowing up, & global proof involves quot schemes. Equivalent version, &

**Corollary:** theorem about compactification of subvarieties of a torus.

\[ R \text{ DVR } = k[[z]] \], field \( k \) of fractions \( k((z,z)) \), \( k = \mathbb{R} \)

\[ X \hookrightarrow \mathcal{G}_m; K \text{ sub variety.} \]

**Lattices**

\[ N = \mathbb{Z}^n \quad (1\text{-param. subgroups}) \]

\[ \tilde{N} = \mathbb{Z}^n \oplus \mathbb{Z} \]

Fan \( \Delta \hookrightarrow \tilde{N} \), with condition \( P_2(\Delta) = \mathbb{Q}_{>0} \) positive ray.

This condition gives us a map of toric varieties.

\[ \text{toric scheme: } \mathcal{Y}(\Delta) \rightarrow \mathcal{Y}(\Delta) \]

\[ \text{Spec} R \rightarrow \mathbb{A}^2 \]; fiber at 0 is a union of toric varieties w/multiplicity

(has origin generic pt.).

Usually \( \text{trop}(X) \hookrightarrow \mathbb{Q}^n \times \mathbb{R}_+ \), but we'll take:

\( \text{Trop}(X) \rightarrow \text{scheme-theoretic closure} \).

**Theorem:** If a fan \( \Delta \) such that \( \overline{X} \) in \( \mathcal{Y}(\Delta) \) is proper over \( R \),

(i.e. special & general fiber are compact), and

\[ G_m^m \times X \rightarrow \mathcal{Y}(\Delta) \]

is flat and surjective. (i.e. \( \overline{X} \) intersects all toric strata -)

Support: In this case, \( |\Delta| = \text{Trop}(X) \). (so possible \( \Delta \)'s are called tropical fans).
Corollary: Take any fan $\Delta$. Then: $\text{Trop}(X) \sim |\Delta| \iff \overline{X}$ is proper over $R$. And $\text{Trop}(X) \sim |\Delta| \iff \overline{X}$ intersects each toric stratum, and the intersection has the right codimension.

Remark: Use the original Bieri-Grives definition of tropicalization, equivariant Raynaud-Gruson, and valuative criterion of properness.

Cor: $\text{Trop}(X)$ is a possibly polyhedral complex of pure dimension $\dim X = d$.

Proof:
1. If $\Delta$ contains a cone $\sigma$ of dim $d+1$, then $\Rightarrow \overline{X} \cap \overline{Y}_\sigma = \emptyset$, contradiction to our map being surjective.

2. Suppose $\Delta$ contains $\sigma$, dim $\sigma < d$. Then:

   $\dim \overline{X} \cap \overline{Y}_\sigma > 0 \Rightarrow \overline{X} \cap \overline{Y}_\sigma$ is affine open stratum, affine.

   $\Rightarrow \overline{X} \cap \overline{Y}_\sigma$ is proper $/ \text{Spec } R$, strictly bigger, $\Rightarrow \sigma$ is a face of a bigger cone.

Example:

![Diagram of a fan and tropicalization]

Normal Fans
$Y(\Delta)$

4 $\mathbb{P}^2$s

special fiber

$\mathbb{P}^2$ $K$

generic fiber

throw away these points

$Laurant series$

$X \hookrightarrow \mathbb{G}^2_m \times \mathbb{K}$ given by Viro polynomial.

Take closure in $Y(\Delta)$, and in this case, this tells us what will happen on a special fiber:

breaks into 4 lines, $\mathbb{P}^1$s.

$X$ family of conics

To do this in the real case, have to take this reflected picture:

4 quadrants in $\mathbb{R}\mathbb{P}^2$

picture of an equator in $\mathbb{C}\mathbb{P}^1$ which describes the degeneration.
Paper is called something like "Hironaka's moving game..."

**Corollary: (Block's moving lemma, Spivakovsky)**

If \( X \hookrightarrow \mathbb{A}^n_k \) is closed, then \( \exists \) an \( n \)-dim. toric variety \( Y(\Delta) \to \mathbb{A}^n_k \) (can choose to be a seq. of blowups of appropriate centers in usual way)

such that \( X_{\text{strict}} \) intersects all strata in right codimension.

\[ \text{closure of } X \cap G \text{ in } Y(\Delta) \]

**Proof:** take any \( \Delta \) supported on \( \text{top} \; (X) \), b apply theorem. \( \blacksquare \)

**Teissier's conjecture:** (char \( k = 0 \))

**Theorem:** If \( X \hookrightarrow \mathbb{P}^n_k \) then one can choose

Veronese reembedding \( X \hookrightarrow \mathbb{P}^n_k \to \mathbb{P}^N_k \) coordinates \( x_0, \ldots, x_N \) +

toric variety \( Y(\Delta) \) of \( G^N_m \) that maps onto \( P^N_k \) such that

\( X_{\text{strict}} \) is smooth.

(Though char \( k = 0 \) b/c at some place this uses Hironaka's rem. of singularities)

Teissier's conjecture is a local (germ) version of this.

**Cor:** (Romer - Shustin)

\[ X \hookrightarrow \mathbb{P}^n_k \] Choose coordinates \( x_0, \ldots, x_n \) such that

intersections of \( X \) with coordinate planes have the right codimension.

\( \Rightarrow \) trop \((X \cap G)\) is a 0-skeleton of the fan of \( \mathbb{P}^n_k \).

4) **Cor:** If \( \Delta \) is a typical fan then \( \bar{X} \) is CM at \( \bar{X} \cap \bigcap_{\Delta} \) \( \approx \)-dim.

**Pf:** use general flatness yoga/ega (bad joke). \( \square \)
Ex: Take any projective $\widetilde{X}$ which is not CM at $p \in \widetilde{X}$.

$\overline{X} \hookrightarrow \mathbb{P}^n$, dim $X = d$.

Choose coordinates generally except make $p$ one of the $D$-dim. strata.

$	ext{drop } (\overline{X} \cap G_m)$ is a $d$-skeleton, but tropical fan must
subdivide this coarsest fan (by above cor).

More precisely, subdivide the top-dim. cone.

Exercise: do this!

Q: Can one tell a fan is tropical just from looking at combinatorial data? No, need to look at $\mathfrak{m}_n$.
Q: (Shrinkelev) any way of turning (t) backwards?

Not clear, true if CM everywhere.