Relationship between $X$ and $\text{top}(X)$:
- describe polyhedral structures on $\text{top}(X)$
- relationship between $\text{top}(X)$ and compactifications of $X$
- how to compute $\text{top}(X)$ without Gröbner bases
- minimal polyhedral structure on $\text{top}(X)$
- geometry behind tropical multiplicities
- how to use tropical tools in elimination theory

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**Flatness**

A coherent sheaf $\mathcal{F}/X$ of $	ext{Spec } A$-module

\[
\begin{array}{ccc}
\text{Spec } A & \xrightarrow{\text{f.t.}} & \text{Spec } B \\
\downarrow & & \downarrow \\
\text{reduced scheme } Y & & \text{B-Noether domain}
\end{array}
\]

$\mathcal{F}$ is called flat over $Y$ if $M$ is a flat $B$-module (tensoring with $M$ preserves exactness).

Geometrically, $\mathcal{F}$ is "continuous" over $Y$.

For example, if $\mathcal{F} = \mathcal{O}_X$, then we say that $X$ is flat over $Y$ if the fibers of $\mathcal{F}$ behave continuously.

**Cheat Sheet:**
- If $X = Y$ then $\mathcal{F}$ is flat $\iff$ $\mathcal{F}$ is locally free.
- Flat morphisms are equidimensional (assuming $X$ is irreducible).
- Under certain circumstances, if $X \to Y$ is equidimensional then it is flat.

For example, if $f$ is projective, and Hilbert polynomial is constant.
• If $X$ is CM and $Y$ is regular (in general, what happens if $X \& Y$ are toric?)
• If $X$ is integral and $Y$ is normal and all fibers are reduced.
• Grothendieck's generic flatness:
  $\exists U \subseteq Y$ such that $F|_U$ is flat.
• If $F$ is flat, then any associated point of $F$ maps to the generic point $\eta \in Y$. (If a local section of $F$ vanishes on $f^{-1}(\eta)$, then it is a zero section).
• If $Y$ is 1-dim. and regular, then the converse is true. (Steps towards "flattening" a sheaf).

**Defn**: The strict transform $\tilde{F}_{\text{st}}$ is a quotient of $F$ by a subsheaf generated by sections that vanish on $f^{-1}(\eta)$.

If $F \to \tilde{F}_{\text{st}}$ has a non-trivial kernel $\implies F$ is not flat.

(*) **Theorem**: (Raynaud-Gruson) There exists a projective birational map $Y' \to Y$ such that $(\tilde{F}_{\text{st}})'$ is flat over $Y'$.

Here, $Y' \leftarrow (g')^* F \rightarrow X' \xrightarrow{g'} X \xrightarrow{\pi} Y' \to Y$

[Claim: understanding the proof might give new arguments in tropical geometry].

**Global proof**: Quot functor $\text{Quot} \left( F/\mathcal{X}/Y \right) = \mathcal{Q}$

$\mathcal{Y}$-schemes $\to$ Sets.

$\mathcal{Q}(S)$ is the set of quotients of the pull-back of $F$ to $S \times_Y X$ which are flat $/$ $S$. 
Suppose \( X \xrightarrow{f} Y \) is projective.

Then \( Q \) is represented by a disjoint union of schemes projective over \( Y \).

By generic flatness, \( F \) is flat over \( U \hookrightarrow Y \), i.e.
\[ Q \rightarrow Y \text{ has a section } s: U \rightarrow Q. \]

We define \( Y' \) as the scheme-theoretic closure of \( U \) in \( Q \).
\[ Y' \rightarrow Q \iff (g')^*F \text{ has a flat quotient, in fact this must be } \]
\[ (\text{deg}')^*F \text{ flat}. \]

**Example:**
\[ \xrightarrow{X = \text{Bl}_0 A^2} F = O_X. \]

\[ \xrightarrow{Y' = A^2} \]

\[ \xrightarrow{\text{Bl}_0 A^2 \cup \mathbb{P}^1 \times \mathbb{P}^1} \text{ one ruling of } \mathbb{P}^1 \times \mathbb{P}^1 \text{ glued to exceptional divisor}. \]

\[ \xrightarrow{E} \text{Bl}_0 A^2 \xrightarrow{\sim} X \]

In this case, \( (O_{X'})_{st} = O_X \) is the main component.
\[ \xrightarrow{\text{iso.}} \text{ so it's flat}. \]
\[ \xrightarrow{Y'} \]

**Conjecture:** \( X \rightarrow Y \) a morphism of algebraic stack
\[ \text{integral, of finite type} / S. \]

\( \text{Noetherian base} \)
\[ \text{of finite type} \]
\[ F \text{-sheaf}. \] Then the Raynaud-Gruson theorem (\#) continues to hold.
Simple case:

Suppose $G$ is a group scheme, smooth affine base scheme $S$. Assume $G \times Y$ is a coherent sheaf on $Y$ (not equivariant). Then, there is a projective birational equivariant morphism $Y' \to Y$ such that the strict transform of $(g^*F)_{st}$ is flat over $Y$, where $g$ is the multiplication morphism

$$Y_s \times G \xrightarrow{m} Y,$$

$$(y, g) \mapsto gy.$$

$$Y' \subset G \xrightarrow{g'} Y \times G, \quad \text{pr}_1^* F = \hat{F},$$

Proof is an exercise. Take previous proof & see everything will work out equivariantly. (Inc Sumihiro's papers!)

If $F$ is a coherent sheaf on $Y$, for example $i_* \mathcal{O}_Z$ for $Z \to Y$. Then there is a class of equivariant blow-ups of $Y$ that flatten $Z' \times G \xrightarrow{m} Y'$, where $Z'$ is a strict transform of $Z$.

If there is a combinatorial way to describe all blow-ups $Y' \to Y$, then we can study spherical varieties (e.g. toric varieties $/k$) actions of complexity $1$ (e.g. toric schemes/DVR).
\[ k, \ S = \text{Spec } k \]
\[ R - \text{DVR}, \ R/m = k, \ S = \text{Spec } R \]
\[ \text{Quot } R = K. \]

\[ G = G_m \]
\[ N = \mathbb{Z}^n \quad \text{toric varieties} \leftrightarrow \text{fans in } N \mathbb{Q} \]
\[ \tilde{N} = N \oplus \mathbb{Z}, \text{ normal toric schemes} / \text{Spec } R \leftrightarrow \text{fans, } \Delta \leftrightarrow \tilde{N} \mathbb{Q} \]
\[ \text{s.t. } \text{pr}_2(\Delta) \subset \mathbb{Q} \mathbb{Z}. \]

\[ \text{if } k \hookrightarrow R \hookrightarrow K \text{ then} \]
\[ Y(\Delta) \twoheadrightarrow Y(\Delta) \]
\[ \downarrow \quad \downarrow \quad \text{toric scheme} \]
\[ \text{Spec } R \longrightarrow A_1^k \]
\[ R \leftrightarrow k[t] \]

uniformizer, \[ \longleftarrow t \]

Take an integral subscheme \( X \hookrightarrow G_K \)

\[ \text{trop}(X) \hookrightarrow \mathbb{R}^n \quad (N, 1) \subset \tilde{N} \]

Let \( \text{trop}(X) \) be a cone over \( \text{trop}(X) \) in \( \tilde{N} \).

Theorem: \( X \) a toric scheme \( Y(\Delta) \) such that \( X \) is proper over \( S \) and \( X \times_S G \rightarrow Y(\Delta) \) is flat and surjective (this we already know).

Then \( |\Delta| = \text{trop}(X) \) and, moreover, if \( Y(\Delta) \) is any toric scheme then

\[ \text{trop } X \subset \Delta \Leftrightarrow \ X \text{ is proper }/S \]

\[ \text{trop } X \supset \Delta \Leftrightarrow \text{the intersection of } \overline{X} \text{ with any toric stratum has the same codimension in } \overline{X} \text{ equal to the codimension of } S \text{ in } Y(\Delta). \]