Last time: \( Y \subseteq T^n \cong (\mathbb{K}^*)^n \), \( \mathbb{K} \) alg. closed field w/ valuation.

\( Y = V(I), I \subseteq \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \)

\[ f = \sum c_i x_i^u \quad \text{trop}(f)(w) = \min (\text{val}(cw) + w \cdot u) \]

\( \text{trop}(V(f)) = \text{locus where trop is not linear} \)

\( \text{trop}(Y) = \bigcap_{f \in I} \text{trop}(V(f)) \)

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**Fundamental Theorem:**

The following sets coincide:

1. \( \text{trop}(Y) \)
2. \( \text{cl}(\{w \in (\text{imval})^n : \text{in}_w(I) \neq < 1 > \}) \)
3. \( \overline{\text{val}(Y)} \)

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**Example:**

\( Y = V(x+y+1) \)

\( \text{trop}(Y) = \)

\( Y = \{(s, -1-s) : s \in \mathbb{K}^* \setminus \{-1\} \} \)

\( \text{val}(s, -1-s) = \begin{cases} (s, 0) & a > 0 \\ (0, a) & a < 0 \\ (0, b) & a = 0 \end{cases} \)

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**Structure Theorem:**

Let \( Y \subseteq T^n \) be an irreducible variety of dimension \( d \). Then \( \text{trop}(Y) \) is the support of a pure \( d \)-dimensional weighted balanced \((\text{im-val})\)-exteral polyhedral complex that is connected in codim one.
Recall: \( W \in (\text{imval})^n \)
\[
\text{in}_w (I) = \langle \text{in}_w(f) \in I \rangle \quad \text{(image in residue field)}
\]
\[
\text{in}_w (f) = t^{-\text{deg}(f) - 1} \prod \sum c_u t^{w_u} x^u \quad W = \text{trp}(f)(w)
\]
\[
= t^{-\text{trp}(f)(w)} \frac{1}{f(t^{w_1} x_1, \ldots, t^{w_n} x_n)}.
\]

Def'n: Given \( I \subseteq K[x_1^x, \ldots, x_n^x] \), let
\[
J = \langle \bar{f} : f \in K[x_1, \ldots, x_n] \cap I \rangle \subseteq K[x_0, x_1, \ldots, x_n]
\]
\[
f = \sum c_u x^u \quad \Rightarrow \quad \bar{f} = \sum c_u x^u x_0 \text{deg}(f) - 1 \quad \text{homogenize}.
\]

Lemma: \( \text{in}_w (I) = \text{in}_{(0,w)} (J) \bigg|_{x_0 = 1} \).

Def'n: Let \( J \subseteq K[x_0, \ldots, x_n] \) be a homogeneous ideal. Fix \( w \in (\text{imval})^{n+1} \).

The Gröbner cell of \( w \) is \( C[w] = \{ w' \in (\text{imval})^{n+1} : \text{in}_{w' \circ (J)} = \text{in}_w (J) \} \)

\[
\begin{align*}
\text{in}_{(3,1)} (x+y+1) &= 1, \\
\text{in}_{(0,0)} (x+y+1) &= 1.
\end{align*}
\]

Prop: \( C[w] \) is an \((\text{imval})\)-rational polyhedron.

The set \( \{ C[w] : w \in (\text{imval})^{n+1} \} \)

forms a polyhedral complex.
Why might this be true? Work through an example:

**Ex:** \( K = \mathbb{C}[[t]] \)  
\( J = \langle tx^2 + xy + ty^2 + x^2 + yz + 3t^2 z^2 \rangle \)

Since \( \text{trop}(y) = \text{cl} \{ w \in \text{ann}(v) : \text{int}_w(I) \neq \langle I \rangle \} \)
\( = \text{cl} \{ w \in \text{ann}(v) : \text{int}_w(J) \} \) does not contain a monomial, \( y^3 \)
\( = \bigcup_{w \in \text{ann}(J)} \langle \langle w \rangle \rangle \)  
which does not contain a monomial

\( \Rightarrow \text{trop}(y) \) is the support of an \( \text{ann}(v) \)-rational polyhedral complex.

**Remark:**
Warning: this polyhedral complex structure is not unique.
(Reason: writing \( T^n \simeq (K^*)^n \) choice, \( \text{trop} \) choice of \( T^n \hookrightarrow \mathbb{P}^n \).

Different choices \( \leftrightarrow \) different Gröbner complexes.

**Exercise:** Compute the Gröbner complex for \( J = \langle 3z^2 + txz - 5tyz + zxy \)  
(as a sanity check, get rid of \( t \)) tropicalize \( + 8t^3 x^2 + t^3 y^2 \rangle \).
Prop: [Bieri-Graber]

Let $Y \subseteq \mathbb{P}^n$ be irreducible of dimension $d$. Then every maximal polyhedron in a polyhedral complex structure on $\text{top}(Y)$ is $d$-dimensional.

Exercise 2: Check this for a hypersurface in $(\mathbb{C}^*)^n$.

Exercise 3: $M_{0,5} = V(z_3 - z_2 + 1, z_4 - z_2 + 1, z_5 - z_2 + 1)$

Compute $\text{top}(M_{0,5})$ (e.g. use Coates).

For picture see Ardila-Klivans, picture of $K_4$.

Weights and Balancing condition

Fix a Gröbner complex structure on $\text{top}(Y)$.

We'll assign a weight $w_\sigma$ to every top-dimensional polyhedron $\sigma \in \Sigma$.

Fix $w_\sigma$ consider $w_\sigma(\sigma)$ for $\sigma \in \text{inv}(\sigma)$.

$V(\text{inv}(\sigma)) \subseteq \prod_{i=1}^n \mathbb{C}$ is a union of torus orbits.

Now count $w_\sigma$ scheme-theoretic multiplicities.

E.g. $f = x^2y + 4y^2 + 3x^2y - xy + 8y + x^2 - 5x^2 + 4$,

$\in \mathbb{C} \{x_i^{\pm 1}, y \pm 4\}$

\[ \text{inv}_0(f) = x^4 - 5x^2 + 4 \]

\[ \text{inv}_{\{2\}}(f) = (x - 1)(x + 2)(x - 2)(x + 2) \]

\[ V(\text{inv}_{\{2\}}(f)) \leq (\mathbb{C}^*)^2 \]

\[ \{(1, 5), (1, -5), (2, 5), (2, -5) : \forall \in \mathbb{C} \} \]

so weight of $(\xi)$ is 4.

Exercise 4: Compute the other three weights and check that we get the right thing.
Note: \(2^{(1)} + 4^{(0)} + (-2)^{(-3)} + (-1)^{(-1)} = (0^{(0)})\)

"zero tension", or "balancing condition."

**Defn:** Let \(\Sigma\) be a \(d\)-dimensional polyhedron.

Formally, \(w_{\sigma} = \sum_{\sigma \in \Sigma} w_{\sigma} \mathcal{H}(P, \text{in}_{\mathcal{W}}(E))\) for associated to \(\text{in}_{\mathcal{W}}(E)\).

**Defn:** Let \(\Sigma\) be a \(d\)-dimensional polyhedron complex. For a polyhedron \(\tau\) of \(d\), \(d-1\). in \(\Sigma\), \(\sigma \in \Sigma\) with \(\tau\) as a face, let \(V_{\sigma, \tau}\) be the primitive lattice vector in \(\mathbb{R}^d / \text{span}(\tau)\) that lies in \(\text{span}(\sigma) / \text{span}(\tau)\).

\(\Sigma\) is balanced at \(\tau\) if \(\sum_{\sigma \in \Sigma} w_{\sigma} V_{\sigma, \tau} = 0\). \(\Sigma\) is balanced if it is balanced at all \((d-1)\)-dim \(\sigma\) faces.

Exercise 5: Check \(\text{tr}p(M_{0,5})\) is balanced.

(For reducible varieties, tropicalize each component separately. Only then we might lose is connected in codimension 1.)