Recall: Yesterday, we spoke about combinatorial patchworking, I used it to construct torus knot curves in the plane:

Claim: better to draw segment like this:

Another claim: better to draw segment in dual plane. More on this shortly.

First:

Generalizations (of patchworking)

1. Higher dimensions

Then: reflect tetrahedron across all axes, and use
an appropriate generalization of toroidal's alg. to extend
distribution of signs (e.g. \((-i, j, k)\) has same sign
as \((i, j, k)\) if \(i\) is even, different if odd, and so on).

Then, to generalize those lines separating

+ - signs as follows:

Draw a vertex at the midpoint of any edge whose endpoints
have different signs, and take the convex hull of these vertices, e.g.
Exercises:

1. Take a primitive convex triangulation of the tetrahedron.
   Calculate the Euler char. of the real surface.
   (Claim: it does depend on the choices we make).
   [Answer: \(-\frac{d^3}{3} + \frac{4}{3} d\)].

2. To find a primitive convex triangulation of the tetrahedron and a disk of signs such that the number of connected components of the surface is equal to the # int. points inside the tetrahedron + 1.
   (N.B. this is not related to maximality in higher dimension, we don't e.g. know what the maximal number of connected components is for a degree 5 curve in \(\mathbb{RP}^3\))
   [e.g. when \(d=5\), a construction with 23 components is known, far more than \# int. points + 1].

Generalizations (cont'd)

(2) Arbitrary Newton polygons (polytopes). What do we mean by that?
Let's stick to dim 2 for the moment:
\[
\begin{align*}
\Delta & \text{ arbitrary} \\
\text{convex subdivision of } \Delta \text{ w/ integer vertices}
\end{align*}
\]

For each sub-polygon, take a polynomial w/ corresponding Newton polygon. Under some conditions these match & can give them together, get a surface whose topology is the same as some piecewise guy constructed from the pieces.

(3) Other properties (critical points, limit cycles of pol. vector fields).
In principle the process can be much more complicated, but should still be able to do such a gluing.

(4) Complete Intersections.

Tropical curves in $\mathbb{R}^2$

$\mathbb{T} = \mathbb{R} \times \mathbb{R}^+$

$\mathbb{T}^\times = \mathbb{R}$

"a \cdot b" = a + b

"a + b" = max $\mathbb{R}, b$.

One can see these tropical operations as a limit of real operations:

Maslov's dequantization of positive real numbers

$S_h$, $h \gg 0$ family of semifields

$\mathbb{R}$, with operations:

$a \ominus_h b = a + b$

$a \ominus_h b = \frac{h}{2} \log (e^a + e^b)$, $h > 0$

Intuition: these are just normal operations

$+$, $\cdot$ on $\mathbb{R}^+$ under the map:

$\mathbb{R}^+ \rightarrow \mathbb{R} = S_h$

$x \rightarrow \frac{h}{2} \log x$

Important fact: these operations are continuous in $h$, $\rightarrow 0$.

Tropical polynomials (in two variables)

"$\sum_{(i,j) \in V} a_{ij} x^i y^j$" = max $\{a_{ij} + ix + jy\}$

piecewise linear convex function, has the outer locus.

This is (almost) the tropical curve defined by our polynomial.
Rectilinear "graphs" (some edges go to $\infty$), each edge has rational slope. Need to put weights on these edges, corresponding to "how singular" our PL function is there.

If $m_1$ has exponents $(i_1, j_1)$, and $m_2$ has exponents $(i_2, j_2)$, assign weight $w_{12}$ to the corresponding edge of the top variety as follows:

$$w_{12} = \text{integer length of this segment}.$$ 

**Example:**

\[
\begin{array}{ccc}
(0,0) & \rightarrow & (1,0) \\
\uparrow & & \downarrow \\
(0,1) & & (1,0)
\end{array}
\]

**Convention from now on:** If the weight is 1, we will omit it.

**More complicated ex:**

One more, all weights are 1.

**Last example:**

This point doesn't matter, so length is 2.
To understand this better, we should subdivide:

there is a natural subdivision of \( \Delta \) such that there is a bijection between subdivision of \( \text{our tropical curve sending faces \rightarrow vertices} \)

edges \( \rightarrow \text{orthogonal edges} \)

vertices \( \rightarrow \text{regions} \)

**Subdivision of the Newton polygon**

\[
\text{Put } \Delta = \text{conv}(V) \\
\text{Consider the following function:} \\
f : V \rightarrow \mathbb{R} \quad \text{graph of our corresponding polynomial.} \\
(i,j) \mapsto a_{ij}
\]

This function defines a convex subdivision of \( \Delta \).

(take the ggpdr graph of the PL function on \( \Delta \)

project down the \( \text{simplicial edges of this graph}. \)

This is convex by definition.

Non-singular tropical curves are those which are dual to prime triangulations. In particular, all the weights are equal to 1.

Tropical curves in \( \mathbb{R}^2 \) can be seen as non-Archimedean amoebas

**Viro polynomial**

\[
\sum_{(i,j) \in E} a(i,j) x^i y^j f(i,j) = \sum_{(i,j) \in E} o(i,j) t^{f(i,j)} x^i y^j
\]

(\( a \) ensures this is a polynomial over \( \mathbb{Q} \).)

\[
(K^*)^2 \xrightarrow{\text{vol}} \mathbb{R}^2
\]
Kapranov's theorem: \[ \sum_{i,j} A_{ij} z^i w^j, \]

the corresponding tropical curve is the inner locus of \[ \max_{(i,j) \in \mathcal{E}} \text{val} A_{ij} x^i y^j, \]

where \( x = -\text{val}_z, y = -\text{val}_w \).

Coming back to our hyperbola:

\[ \text{Correspondence between arch areas here}. \]