"Tropical geometry are more than shadows of classical objects."

Basic notions:

1. Tropical semi-ring \( T \) is \( \mathbb{R} \cup \{-\infty, +\infty\} \) with \( +, \cdot \) rules instead of max.

\( x + y = \max \) \( x^y = x + y \) \( O(T^*) \) (almost constant sheaf, but some poly's, e.g. \( x^{-1} \) are not defined at \( -\infty \)).

Draw picture:

\[ T^n = [-\infty, +\infty]^n \]

Affine line:

We can study \( O(T^n) \) as well:

- \( O \) is determined by \( \mathbb{Z} \)-affine structure on \( \mathbb{R}^n \subset T^n \)

\( \text{some we have a well-defined integrable notion of integer tangent vector.} \)

In addition to regular functions, we may want to consider rational functions

\( \frac{f}{g} \). Again we can build a sheaf out of them (Euclidean topology as before).

Each \( f \in O(U) \) defines a hypersurface, where \( \frac{1}{f} = -f \) is not locally regular.
Example: a rational function in \( \mathbb{T} \)

\[
\text{poles}
\]
\[
\text{zeros}
\]
\[
\text{all steps are integers.}
\]

If we consider the divisor of \( a + bx + cy \)

\[
\text{"abc"} = 0_{\mathbb{T}}
\]

This should also be an ellipse-like, but looks different from \( \mathbb{T} \)!

Any regular function \( f \in \mathcal{O}(U) \) defines its principal open set \( D_f \)

\[
f(x) = "x+a"
\]

this is the set theoretic graph, hence \( f \in U \).

But consider tropical completion,

look at "\( y + f(x) \)" fills in dotted line, with \(-\infty\):

In the Zariski topology,

\[
D_f \rightarrow \text{"tropical modification along } V_f\text{"}
\]

Equivalence means differ by maps that locally look like step modifications.
To rephrase:

Define tropical modification. Given \( f \in C(U) \), consider

\[ U \times \mathbb{T}, \text{ and with its sheaf of regular \( C(U \times \mathbb{T}) \) \text{ look at \( y + f(x) \)}} \]

\[ \Gamma_{y + f(x)} = \tilde{U} \subset U \times \mathbb{T} \]

This map \( \Gamma \) is called tropical modification along \( f \).

Two important features of this. Suppose we have

[Diagram of a chart here, homo to a subset of \( \mathbb{R}^n \) homeo to an equivalence].

Then:

- \( \tilde{U} \subset \mathbb{T} \), polyhedral complex of full dim. n.
- \( \tilde{U} \) satisfies the balancing property on codim-1 skeleton.

Balancing condition:

\[ \sum v_i = 0 \]

(Smooth)

Tropical manifolds in the narrow sense:

- locally given, by modifications of \( U \subset \mathbb{T} \) with smooth centers which are
- tropical manifolds in the narrow sense, in dimension \( n - 1 \).

And a single point or \( \emptyset \) is smooth.

Examples: \( \dim 1 = \text{codim} 1 \).
\[ TP^n = \bigcup_{n+1} \]  

Hypersurfaces in \( TP^n \)  

primitive triangulations

(\text{In dinc. 1, phylogenetic trees --- just get finite graphs})

compactness & balancing criteria \( \rightarrow \) all edges leaves adjacent to 1-valent vertices have \( \infty \) length.

Get length from \( \mathbb{Z} \)-alpha scheme.

c.e.g.

Grassmannians + Flag varieties?

\( \text{We'll return to } G_{\mathbb{R}_n} \text{ when we relax our def'n of topological manifold more.} \)

Remark by audience member:

\[ \text{is smooth, but it's not a hypersurface? (why?)} \]

\( \text{The only smooth hypersurfaces arise from primitive triangulations} \)

\( \text{These things can locally be written as a limit of complex analytic manifolds.} \)

\( \text{One more example: abelian varieties} \)

in \( \mathbb{R}^n \times TP^n \)

typical torus
One condition for this torus to be projective: can find words for this so that 
first word of \( v_2 \) = second word of \( v_1 \), etc.

Maps \( T^n \rightarrow \mathbb{R}^n \rightarrow \mathbb{R}^n \rightarrow \mathbb{R}^M \) 
are locally given by affine map with \( \mathbb{Z} \)-linear part.

These are locally given by affine map \( \mathbb{R}^n \rightarrow \mathbb{R}^n \) of \( \mathbb{Z} \)-linear part.

Example:

\[
\begin{array}{c}
\downarrow \\
\text{scale by } 2
\end{array}
\begin{array}{c}
\downarrow \\
\text{isometric}
\end{array}
\]

This is a tropical map, tropicalization of \( \mathbb{R}^1 \rightarrow \mathbb{R}^2 \)

(need to scale one branch by 2 to preserve balance, condition on primitive vectors).

Example:

\[
\begin{array}{c}
\downarrow \\
\text{elliptic curve branched at 4 points.}
\end{array}
\]

Remark: integrality of maps required because we want to see integer-exponent polynomials.)
Another example:

Build a K3 surface:

24 special points on each prism, get Z-affine structure on complement of points.

(ref. Kaledovich-Soibelman, Gross-Siebert)

Given $X^k \rightarrow \mathbb{TP}^N$, can define degree as intersection with $(n-k)$-face of simplex.

Now, $\tilde{U} \subset T^N$ enlarge regular functions $f$ if it comes from a rational function in $T^N$ and convex on (some 1 skeleton). (Heavily uses get a balancing condition in codim 2 only.)
The resulting objects are described by nefroides. Sometimes these objects are not locally pliable, or there's an obstruction to finding global phases, even for curves.