"Tropical varieties are combinatorial shadows of classical varieties."

\[ T = (K^*)^n, \quad K \text{ field, } K^* = K \setminus \{0\} \]

Study \( Y \subseteq T \) subvariety

\[ Y = \mathcal{V}(I) \] ideal in \( K[x_1^\pm 1, \ldots, x_n^\pm 1] \]
\[ \{ x \in T^n : f(x) = 0 \text{ for all } f \in I \} \]

Goal: \( Y \subseteq T^n \xrightarrow{\text{tropical}} \text{trop}(Y) \subseteq \mathbb{R}^n \)
alg. geometry \quad polyhedral combinatorics

Goal: Compute invariant of (compactifications of) \( Y \) from \( \text{trop}(Y) \).

\[ \text{Tropical philosophy} \]
\[ + \quad \rightarrow \min \]
\[ * \quad \rightarrow + \]

Q: Why do we care about subvarieties of tori?

A1: This is somehow where tropical geometry works

A2: There are interesting varieties there.

E.g. hyperplane arrangement complements.

\[ A = \{H_a, \ldots, H_s\} \subseteq \mathbb{P}^N \]
\[ Y = \mathbb{P}^n \setminus \bigcup_{i=1}^s H_i \]

Claim: \( Y \hookrightarrow T^{s-1} \).

Choose \( a_1, \ldots, a_s \in \mathbb{C}^{s+1} \)

\[ H_i = \{ x \in \mathbb{P}^n : a_i \cdot x = 0 \} \quad \text{with} \quad (x_0 : \ldots : x_n) \]

\[ A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \]
\[ b_i = [-1 \ 1 \ 1 \ 0] \]
\[ Y \rightarrow T^{n-3} = (\mathbb{C}^*)^{n-3} / \mathbb{C}^* \]

\[ \times 1 \rightarrow (x \cdot a_1, -x \cdot a_3) \]

Choose \( b_1, \ldots, b_{n+1} \) a basis of \( \text{Ker}(A) \)

\[ I = \langle f_i \rangle \]

\[ f_i = K[z_1^{\pm 1}, \ldots, z_{n+1}^{\pm 1}] \]

\[ f_i = \sum \{ b_{ij} z_j \} \]

Claim: \( \forall (I) \leq (\mathbb{C}^*)^{n-3} / \mathbb{C}^* = Y \)

**Important special case**

\[ M_{g,n} = \text{moduli space of smooth genus } 0 \text{ curves with } n \text{ distinct marked points} \]

\[ = (\mathbb{P}^1)^n \setminus \text{diag} / \text{Aut}(\mathbb{P}^1) \]

\[ = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \text{diag} \]

\[ = (\mathbb{C}^*)^{n-3} \setminus \{ x_i = 1 \} \]

---

**What is \( \text{trop} (Y) \)?**

**First example**

\[ Y = V(f), f = \mathbb{C} [x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \]

\[ e.g. \ Y = V(x + y + 1) \leq (\mathbb{C}^*)^2 \]

\[ f: \mathbb{C}^n \rightarrow \mathbb{C} \]

\[ \text{trop}(f): \mathbb{R}^n \rightarrow \mathbb{R} \]

\[ \text{trop}(f)(w) = \min_{C_w \neq 0} w \cdot u \]

\[ (1, 0, 0, 0) \]

\[ e.g. f = x + y + 1 \]

\[ \text{trop}(f)(w) = \min (w_1, w_2, 0) \]

\[ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]
\[ \text{trop} (V(f)) = \text{region where trop}(f) \text{ is not linear}. \]

**Generally**

K algebraically closed field with a non-trivial valuation, i.e.
\[ \text{val} : K \rightarrow \mathbb{R} \cup \{\infty\} \]

s.t. 1) \( \text{val}(a) = \infty \) iff \( a = 0 \)

2) \( \text{val}(a \cdot b) = \text{val}(a) + \text{val}(b) \)

3) \( \text{val}(a+b) \geq \min(\text{val}(a), \text{val}(b)) \).

**e.g.:**

* K = \( \mathbb{C} \{\{t\}\} \) Puiseux series
  \[ = \bigcup_{n \geq 1} \mathbb{C}\{\{t^{1/n}\}\}\) Laurent series

\( \text{val}(a) = \text{lower exponent occurring} \).

**e.g.:** \[ a = 3t^{-1/2} + 7t^2 + \pi t^{100} + \ldots \]
\[ \text{val}(a) = -\frac{1}{2}. \]

**Q_p, p-adics.**

**R valuation ring of K**

\[ \mathfrak{m} = \{a \in K; \text{val}(a) \geq 0\}, \text{ with maximal ideal} \]
\[ M = \{a \in K; \text{val}(a) > 0\} \]

[note: \( \text{val}(\frac{1}{a}) = -\text{val}(a), \text{val}(1) = 0 \), so if \( \text{val}(a) = 0 \), a is a unit] .

**k residue field, R/m**

**e.g.:** \( K = \mathbb{C}\{\{t\}\}, k = \mathbb{C} \)

\( K = \mathbb{Q}_p, k = \mathbb{Z}/p\mathbb{Z} \).
For \( f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \), define
\[
\text{trop}(f) : R^n \rightarrow R
\]
\[
\text{trop}(f)(w) = \min_{u \in Z^n} \left( \text{val}(c_u) + w \cdot u \right) \quad (\text{where } f = \sum_{u \in Z^n} c_u x^u, c_u \in K) \\
\text{val}(c_u) \rightarrow \min \quad \text{piecewise linear function.}
\]
\[
x \rightarrow \min
\]

Now define \( \text{trop}(V(f)) = \text{locus where trop}(f) \text{ is not linear} \).

\( f = tx^2 + ty^2 + xy + x + y + t \in \mathbb{C}[[x^{\pm 1}, y^{\pm 1}]] \)
\[
\text{trop}(f) : R^2 \rightarrow R
\]
\[
\text{trop}(f) = \min \{ 2x+1, 2y+1, xy, x, y, 1 \}
\]

Exercise: Carry this out for \( f = t^2x^3 + x^2y + xy^2 + t^2y^3 + x^2 + t^{-1}xy + y^2 + x + y + t^2 \).

\[
Y = \{ x \in T^n : f(x) = 0 \text{ for all } f \in I^3 \}
\]
\[
V(I) = \bigcap_{f \in I} V(f)
\]
\[
\text{Def: trop}(Y) = \bigcap_{f \in I} \text{trop}(V(f)) \quad \text{in this case \ it \ often \ does \ not \ suffice \ to \ take \ a \ generating \ set \ for \ I!}
\]
Questions
1) why this def'n?
2) How do we compute trop(Y)?

Start with #2 above $\mapsto$ Gröbner theory

Notation: $a \in R$ (val(a) $\geq$ 0)

$\bar{a}$ image in $K = R/M$.

Choose a homomorphism $t: \text{val} \rightarrow K$ splitting valuator. (This is a choice, may require Noetherian hypothesis??).

For $w \in (\text{im} \text{val})^n$, $f = \sum c_u x^u$;

let $W = \min_{u \in \mathbb{N}^n} (\text{val}(c_u) + w \cdot u) = \text{trop}(f)(w)$

$\downarrow$ Image $b$ in $R/M$.

Define $\text{in}_w(f) = t^{-W} \sum c_u t^{\text{val}(c_u)} x^u \in K[x_1^\pm 1, \ldots, x_n^\pm 1]$

$\uparrow$

$\downarrow$ takes a polynomial with coefficients in $K$, vector, returns poly. w/ coeff in $K$.

$\text{ex.}$ $w = (0,0) \rightarrow W = 0$

$\text{ex.}$ $f = tx^2 + ty^2 + xy + x + y + t \in \mathbb{C}[x,y]$; $w = (0,0), W = 0$

\[ \text{in}_{(0,0)}(f) = xy + x + y \]

\[ \text{in}_{(2,2)}(f) = t^{-1}(t^2y^2 + t^2y^2 + t^3x^2 + t^2x + ty + t) \]

$= y + 1.$
Def: \( Y = V(I), \; I \subseteq K[x_1^\pm 1, \ldots, x_n^\pm 1] \)

\( W \in (\text{imval})^n \).

Initial ideal \( \text{ini}_w(I) = \langle \text{ini}_w(f) : f \in I \rangle \)

Again, can't just use generators of \( I \), but there is some finite set.

Note: \( \text{ini}_w(f) \) is not a monomial iff \( \text{ini}_w(f)(w) \) is achieved twice so \( \text{ini}_w(f) \) is not linear at \( w \), so \( w \not\in \text{top}(V(f)). \)

So \( \text{ini}_w(I) \) is not the unit ideal \( \implies w \not\in \text{top}(V(f)). \)

**Fundamental theorem of tropical algebraic geometry:**

Let \( Y \subseteq \mathbb{T}^n, \; Y = V(I), \; I \subseteq K[x_1^\pm 1, \ldots, x_n^\pm 1] \).

Then the following sets coincide:

1) \( \text{top}(Y) = \bigcap_{f \in I} \text{top}(V(f)) \)

2) \( \text{closure in } \mathbb{R}^n \) of \( \{ w \in (\text{imval})^n : \text{ini}_w(I) \neq \langle 1 \rangle \} \)

3) \( \text{val}(Y) = \text{closure in } \mathbb{R}^n \) of \( \{ (\text{val}y_1, \ldots, \text{val}y_n) : Y = (y_1, \ldots, y_n) \in Y \} \)

(*\#3 justifies the "shadow of classical variety" remark).

Notes: 1) = 2) is easiest.

If \( \text{ini}_w(I) = \langle 1 \rangle \), then you weren't in \( \text{top}(V(f)) \), and (a).

3) \( \leq 1) \) is easy.

1) \( \leq 3) \) is elementary but tricky.

[Kapranov, Sturmfels, Speyer - Sturmfels, ...]

*(How do we compute this? #2) above helps.)*
(2) tells us how to compute $\text{tr}_{Y}$

     google: gfan, jensen

(3) justifies the def'n

"Tropical varieties are combinatorial shadows of classical varieties."

Tomorrow: More about tropical structure, more examples.

Answer to exercise at beginning:

See attached exercises for lecture 1.