What is an amoeba?

Image of an algebraic variety under a coordinate-wise valuation map.

So non-archimedean amoebas = tropical varieties.

We consider Archimedean (complex) amoebas associated with the map $C^* \mapsto \log |z| \in \mathbb{R}.$

Have to deal with the triangle inequality $|z+w| \leq |z|+|w|$ (rather than $|z+w| \leq \max(|z|,|w|)$).

Background: G. Bergman '71: "Logarithmic limit sets"  
O. Viro  
Gelfand-Kapranov-Zelevinsky: '94 introduced the term amoeba.

Plan

I. Study amoebas as the link between complex, real, and tropical geometry  
II. Explore the duality between amoebas and coamoebas.

The case of a line:

At $1+z+w=0$ the complex  

or

special points: intersects

$w \times \bar{w}$ was $800.$

tropical.
Component-wise modulus in real (case):

complex case:

$1 \leq |z| + |w|$

$|z| \leq 1 + |w|$

$|w| \leq 1 + |z|$

topologically, pull out in three directions & flatten.

There will always be a tropical curve inside the amoeba describing its topology, but may be more complex! Also, relation between real/complex curves in general is more complicated.

(Amoebas arise naturally in complex analysis when trying to expand rational functions into Laurent series).

Laurent Series of rational functions $(n = 1)$

$f(z) = (z-1)(z+2)$

Q: What are the Laurent series expansions of

$\frac{1}{f} = \frac{1}{(z-1)(z+2)} = \frac{1}{3} \frac{1}{z-1} - \frac{1}{3} \frac{1}{z+2}$

Answer: there are three:

$x = -\frac{1}{3} \sum_{k=0}^{\infty} z^k - \frac{1}{6} \sum_{k=0}^{\infty} \left( \frac{z}{2} \right)^k$

In higher dimensions, you won't always get such a nice factorization.
Alternatively, geometric series at once!

\[
\frac{1}{z^2 + z - 2} = -\frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{z^2 + z}{2}\right)^k, \quad \text{for } 1|z^2 + z| < 2.
\]

\[
= -\frac{1}{2} - \frac{1}{2} \frac{z^2 + z}{2} + \ldots \quad \text{ actually ok on bigger disc.}
\]

(Then it gets a bit more complicated... can try to expand using \( z \) but then only get convergence on \( \mathbb{C} \)).

**Def**: A polynomial \( f \) is ruled (lopsided) at \( z_0 \) if one of its monomials has larger absolute value than the sum of the absolute values of remaining monomials. This helps us solve (ref. "cyclic resultant" in one variable case).

Let \( f \) be a Laurent polynomial \( f \in \mathbb{C}[z_1^{\pm}, \ldots, z_n^{\pm}] \).

\[
f(z) = \sum_{\alpha \in \mathbb{Z}_+^n} a_{\alpha} z^\alpha, \quad A \subset \mathbb{Z}_+^n \text{ finite, } \mathbb{Z}_f = \{ z \in \mathbb{C}_*^n \mid f(z) = 0 \}
\]

Newton polytope \( \Delta_f = \text{conv} (A) \cap \mathbb{R}^n \) convex hull.

\[
\hat{\mathbb{Z}}_f = \{ s \in \mathbb{C}_*^n \mid f(e^s) = 0 \}
\]

\[
\mathbb{R}^n \xrightarrow{\text{Exp}} \hat{\mathbb{Z}}_f \xrightarrow{\text{Im}} \hat{\mathbb{Z}}_f \xrightarrow{\text{Re}} \mathbb{R}^n \xrightarrow{\text{Arg}} (\mathbb{R}/2\pi\mathbb{Z})^n
\]

\[
\text{Log} = (\log |z_1|, \ldots, \log |z_n|)
\]

\[
\text{multi-valued, } \text{Arg} (z) = (\arg z_1, \ldots, \arg z_n).
\]
Def: The Amoeba $A_f := \text{Log}(Z_f)$
Cor-amoeba $A'_f := \text{Arg}(Z_f)$.

Properties:

$A_f$ is closed (b/c fibers of Log are compact). $A'_f$ not necessarily.
(\text{in\,general,\,not}).

$A'_f = \bigcup_{r \in \Delta_f} A'_f$, face of any dimension

Ex: $f = 1 + z + wz$

\[
\Delta_f \quad \text{Looks at}
\]

The connected components of $R^n \setminus A_f^{(1)}$ are convex
(careful what we mean about that -- $A_f^1$ is on the faces, not on $R^n$).

Thm [Forster - Passare - Tsikh]: There is an injective map

\[
\bigcup \text{components of } R^n \setminus A_f \xrightarrow{
u} \Delta_f \times \mathbb{Z}.
\]

The recession cone of a component $E$ is equal to the normal cone of $A_f + t\text{translated}$

\[
\gamma(E) := \left\{ x \in R^n : \langle \nu, x \rangle = \max_{\alpha \in \Delta_f} \langle \alpha, x \rangle \right\}
\]

$C_{\gamma} = \left\{ x \in R^n : \langle \nu, x \rangle = \max_{\alpha \in \Delta_f} \langle \alpha, x \rangle \right\}$
because every vertex of $\Delta_f$ is achieved by $v$. \\
\text{Cor:} \quad \# \text{vert. } \Delta_f \leq \# \text{components of } \mathbb{R}^n \setminus \Delta_f \leq \# \Delta_f \cap \mathbb{Z}^n \\
\text{What can we have in between?} \\
\text{Thm: (Rullgard) Any intermediate configuration can occur.} \\

\textbf{Homework:} \\

1. \[ \begin{array}{c}
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\text{What are the possible values of } (x,y) \text{ and } (u,v) \\
\text{Compute the Jacobian } \frac{\partial (x,y)}{\partial (u,v)}, \text{ try to explain it.}
\end{array}
\end{array} \]

2. Draw the amoeba of \\
\[ f(z, w) = z + z + w + z \cdot w \]

3a. Determine for which $a \in \mathbb{C}$ the amoeba of $f(z) = 1 + az + z^2$ has a component of order 2.

3b. \[ f(z,w) = 1 + az + z^2 + w^3 \quad \text{order (1,2)} \]