joint work (in progress) with C. Wendt, K. Niederkrüger

Motivation: want some rough classification of contact structures in higher dimensions.

\((V, \xi)\) closed, connected contact manifold

Def: \((W^2, \omega)\) comp. symplectic manifold is a filling of \((V, \xi)\) if
\[\exists W = V \text{ and there exists a vector field } Y \text{ near } V \text{ s.t. } L_Y \omega = \omega,\]
\[\omega|_V \text{ is a contact form for } \xi.\]

(Filling gives you augmented, cht, can linearise, nice structure, etc.)

\((A, \partial)\) a differential graded algebra over some ring \(R\)

An augmentation of \((A, \partial)\) is a unital dga morphism of degree 0.
\[\varphi: (A, \partial) \to (R, 0), \text{ (this version requires } W \text{ exact; in general } R \text{ depends on } W)\]

i.e. \(\varphi: A \to R\) is an algebra map s.t. \(\varphi(1) = 1, \varphi \partial = 0\)

(overtwisted) manifolds are not fillable can be seen by \(\text{CH}: \text{CH} = 0, \text{ so } \nexists \text{ augmentation, i.e. } \nexists \text{ filling.}\)

Review of SFT:
\((V, \xi)\) closed contact mifold, \(A\) non-deg. contact form.

\(A\) free graded comm. alg. generated by the good closed orbits over \(R (= \mathbb{Q} \text{ for now}), \mathbb{Z}/2\mathbb{Z} I, (2/1 \text{ grading now; grade by parts of } u -1 + \text{comm. degree}).\)

\(d: A \to A \text{ counts } \int 1 \text{ intersection points } \in R \times V\)

contact homology
\[H^\ast_c (V, \xi) = H^\ast_c (A, \partial)\]
(even stranger, https type of this dga should be an invariant)

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Higher genus

Algebraically, these can be combined into one map

\[ D : A [[ t ]] \rightarrow A [[ t ]] \]

s.t.
1. \( D D = 0 \)
2. \( D(1) = 0 \)
3. \( D = \sum_{k=1}^{\infty} D_k + \frac{1}{k!} \), where \( D_k : A^* \rightarrow A^* \) is a differential operator of degree \( \leq k \).

(\( A \) is a poly algebra)

\( \tilde{t} \) keeps track of genus + #pos punctures.

In general, \( D_k \):

- highest order + lower order (higher genus)

Today, not keeping track of marked points, homology classes, etc.

Just rigid curves

The full (rigid) SFT of \( (V, \xi) \)

\[ H^{\text{SFT}}_*(V, \xi) = H_* \left( A[[\hbar]], D \right) \]

Warning: *this is not an algebra (\( D \) can have arbitrarily high order)*

* \( D \) is \( \hbar \)-linear, i.e. \( D(\hbar \cdot Q) = \hbar \cdot D(Q) \)

\[ \Rightarrow H^{\text{SFT}}_* \text{ has the structure of } \mathbb{R}[[\hbar]] \text{-module}. \]

* \([A] \in H^{\text{SFT}}_* \text{ always exact, and thus so does } [\hbar] \)
Def: \((V, \mathcal{F})\) has algebraic torsion of order \(k\) if \([t^k] = 0\) in \(H^S_{\text{FT}}(V, \mathcal{F})\).

Prop: (1) If \((V, \mathcal{F})\) has algebraic torsion of some order, then it's not fillable.

(2) If there exists an exact symplectic cobordism from \((V', \mathcal{F}')\) to \((V, \mathcal{F})\), and \((V', \mathcal{F}')\) has torsion of order \(k\), then so does \((V, \mathcal{F})\).

Pf: Let \(RW\) be the Novikov completion of group ring of \(H_2(W; \mathbb{Z})/\ker W\) w/ coeffs. in \(R\). Denote by \(A_W^- = A^- \otimes_R RW\).

The cobordism gives rise to two structures:

(a) \(\Phi: \mathcal{X}^W \to \mathcal{X}^W \mathcal{D}W = \mathcal{D}^-W\)

(b) \(\Phi : (X^+, D^+) \to \mathcal{X}^-W \mathcal{D}^-W\)

\(\Phi\) is a chain map, where \(\mathcal{D}W(Q^-) = e^A D^-(e^A Q^-)\).

For \(W\) a filling, \(V^- = \emptyset \Rightarrow \mathcal{X}^-W = R W[[t]]\)

\(D^- = 0 \Rightarrow \mathcal{D}^-W = 0\).

\(\Phi(1) = 1\).

\(\Phi(t^k) = t^k\), so \(m\) isn't have tors...

Examples:

Algorithm in dim 3.
\( k = 0 \iff \text{alg. overtwisted}, \text{i.e. } H_n(X, \mathbb{R}) = 0 \iff \text{overtwisted} \)

\( k = 1 \iff \text{Giroux torsion} \)

**Construction for general \( k \):**

Pick \( V_g = \Sigma_g \times S^1 \). Pick \( \Gamma \subset \mathbb{R} \) collection of \( 1 \leq k + 1 < g \) simple closed curves \( \Gamma \subset \Sigma_g \). \( \Gamma \) lifts \( \Sigma_g \) into \( 2 \) components, one of which has genus \( 0 \).

\[ \begin{align*}
\text{Exist unique ambient structure in } V_g \text{ at all } \Sigma_g \text{ pts. conv. surfaces/ } \Gamma \text{ as branch set. How to orient? something involving genus 1 side.} \\
\text{gives me an } S^2 \text{-family of orbits/braids/circles for } 2
\end{align*} \]

Alternatively: Start w/ open books w/ \( k + 1 \) binding orbits and trivial monodromy. One has page of genus \( 0 \), we have higher genus. Blow up the binding/give open book structure another one, the result together. (Wendl's "summed open book") — claim that you get the same contact manifold.

\( \exists \jmath \text{ s.t. pages are holomorphic of index } 2 - 2g \). Claim no other.

(Recall \( D \) of the form \( \sum_{k \geq 1} D_{k+1} \frac{h^k}{k} \))

Next rigid disks \( \to \text{tame at order } k \). Need to argue they don't have form of lower order — not quite written up yet.

These will be examples that have exactly \( k \) tame but not \( (k-1) \)-tame.

Hence: In general, if you have at least \( 2 \) regions separated by dividing set is a-blow

\( \text{ex, it's not fillable. (bc Heegaard Floer contact invariant } \mathbb{Z}/2 \text{ vanishes) } \)

(Also, \( \text{Wendl can prove that ECH-contact invariant we see } K \text{ vanishes here) } \)

In those examples, \( \exists \theta \in \mathbb{R} \text{ s.t. } \exists ! \theta \text{ curves asymptotic to some subcollection of those. } \)

(\( \theta \) curves asymptotic to some subcollection of those.)

of any genus.