• Statement of the Problem

• Discussion of group actions (on categories)

• Attempted solution & why it fails

• Twisted Koszul Duality & Brauer Group of $K^*$ ?

• Geometric Proposal

$X$ compact symplectic manifold,
$G$ compact group, Hamiltonian action on $X$

Goal: Define a gauged TQFT (gauged GW-theory)

• Defined on (convergent flat $G$-bundle)

• Can integrate out gauge field

• GW theory of $X//G$ as an "adiabatic limit" (application)

"Known" for some time that it's possible.

Topologically meaningful: $K$-theory

$H^*$: large level limit.

Reason: $X$ compact manifold, $K_G(X)$ has something like PD, $H^*_G(X)$ does not.

"Pure" gauged theory: $*/G$ must be regularized

Analogy: mirror to non-compact Calabi-Yau $(G_6/G_6^{(c)})$, need LG potential to compactify

(Frenkel, Toledano-Laredo, -)

Gauging is "easy" for open-closed TQFT, include the category of boundary states.

Need: $G$-action on $F(X)$

"Suffices" to know the equivariant mirror.

Recall: (Costello, Kontsevich, Hopkins-Lurie)

* open-closed TQFT for smooth surface $\leftarrow$ category

* extension to nodal curves $\leftarrow$ obstruction, extra datum if lifted
\( \mathcal{C} \) has \( G \)-action:

\( \rightarrow \) Category of gauged boundary states "controlling" the gauged theory

normally:

- \( \cdots \rightarrow \mathcal{C} \).

For us:

\( \mathcal{G} : \) "locally system of categories over \( BG \)" (\( G \) finite).

\( \mathcal{E}G = \mathcal{R} \Gamma (BG; \mathcal{C}) \) should be the gauged category for a point.

E.g.: \( \mathcal{C} = \text{A-mod}, \ G \text{ acts on } A \text{ by auto} \), E.g. \( A = \mathbb{C} \)

(\( G \) finite) \( \mathcal{E}G = G \times \text{A-modules} \)

\( \text{get} \ (\text{Vect}) \ G = \text{Rep} (G) \)

(need to explain "eigenvalues Frobenius" but we won't today.)

When \( G \) compact Lie,

need "differentiable" \( G \)-acts on \( \mathcal{C} \),

(try) translation of the Lie algebra action

\[ G \hookrightarrow \mathcal{C}, \text{ then } \phi \xrightarrow{L} \text{Der} \ (\mathcal{C}) = \text{Lie alg. map} \]

\[ \text{translation: } \tilde{\iota} : g \rightarrow \text{HCH}^0(\mathcal{C}) \quad [\beta, \tilde{\iota}] = \mathcal{C} \]

\( \tilde{\iota} : g \rightarrow \text{HCH}^0(\mathcal{C}) \) (analogue of a Hamiltonian action)

\( \tilde{\iota} \) : 1-form on \( B(H) \) (internal degree), so this above thing is like a connection, so which is what we mean by "local system"

E.g.:

(i) \( X \) compact manifold, \( \mathcal{C} = (\text{\Omega}_X^\ast, \partial) \)-mod

Have \( \mathcal{L} : \phi \rightarrow \text{Vect}(X) \text{ Lie alg. hom} \).

\[ \tilde{\iota} : \phi \rightarrow \text{HCH}^0(\mathcal{C}^\ast X) \]

\[ \tilde{\iota} \rightarrow \tilde{\iota}(\tilde{\gamma}) = \mathcal{C} \tilde{\gamma} \] (ii) \( X \) compact symplectic, \( G \)-action Hamiltonian, \( m : X \rightarrow \phi^\ast X \)

\( L \rightarrow \phi \) If \( \phi = \exp (\tilde{\gamma}) \) small \( \tilde{\gamma} \), construct iso \( \phi (\phi L, V) \sim (L, V) \)

\( \text{Lagr}(X) \rightarrow \text{Loc}(L) \)
Note: the trivialization of the action is integrable.

If $G = T$, $T$ acts, $\exp(\mathcal{L})$ acts trivially, get action of $T/\exp(\mathcal{L}) = B\pi_1(T)$,

$B\pi_1(T)$ acts on $\mathcal{F}(X)$,

functions on $T^\vee$,

$$\pi_1(T) \to \text{Aut}(\mathcal{F})$$

$\mathcal{C}[\pi_1(T)] \to \text{HH}^0(\mathcal{F})$ \Rightarrow \text{fibers over } T^\vee

(ie. $\text{HH}$ is an algebra over $\mathcal{C}[\pi_1(T)]$).

Known consequence:

B-model: Mirror $X^\vee$ of $X$ comes w/ a hol. map to $T^\vee$!

(construct equivariant mirrors)

(general, want a graded version)

(ii) $X$ symp, $A = \text{def. quant. of } X$,

$\mathcal{H}: \text{U}_q \to A$ alg. hom., s.t. $[\mathcal{H}(g), \cdot] = \mathcal{H} g$

$\mathcal{C} = A - \text{mod}$ (perfect)

$\text{HH}_X A = \text{HH}^0 X$, but $\mathcal{C}$ doesn't see any hol. diff, it does see some Leg/hs.

(iv) $A = U(g)$ $G$-actin

Rep$(g)$

we'll see: gauged category is $\text{Rep}(G)$.

(assume our categories are modules over an algebra)

Write a proposal for $\mathcal{C}(G/G)$ (assume $\mathcal{C} = A - \text{mod}$)

Observe: data gives an action of the supergroup $(G \times 0_G, 0_G)$ on $A$

Can form crossed product $C(G) \otimes \Lambda\otimes A$.

$C_c(G \otimes \Lambda) \otimes A$

$C_c(G \otimes \Lambda) \otimes A$
\[ A = C, \ \text{get} \ (C_x(G), \delta) - \text{mod} \ \text{w/ Porteraggi,} \]
\[ H_x(G) \quad (\text{formal}) \]

\[ A = U(g), \ \text{get} \ C(Ca), \ \text{convolution} \ \text{Rep}(G) \]

This is the 'wrong' model, gives a localization of the category.

Correct answer: "Cartan model"

(1) \[ G \wedge (\text{Func}(g) \otimes A) \]
\[ \rightarrow \quad \check{\mathcal{G}}_a \] basis of \( g \)
\[ \check{\mathcal{G}}_a \] basis of \( g^* \)

This really means:
\[ \check{\mathcal{G}}_a(\check{g}) = dA = d = m_a \]

"\( \mathcal{C}^0(g) \otimes \)"
\[ \check{\mathcal{G}}_a \cdot \check{\mathcal{G}}_a = W = m_0 \]
\[ d^2 = [W, \cdot] \]

(Can't use \( \mathcal{C}[[g^*]] \) b/c you lose information for \( \text{Func}(g) \))

Partial K-theoretic answer, top meaningfull:

\[ \text{Func}(g) \mapsto \text{Func}(G) \]

Claim: the whole thing makes sense despite multi-valuedness of the \( \check{\mathcal{G}}_a \).

From now on,

\[ \mathfrak{X} \to C \quad \text{gauge theory of a point} \]

Finiteness of the theory requires a regularization on \( (g^*) \)

(when \( A = C \), this is some sort of B-model mirror of \( B \in a \))

Twisted Koszul Duality:

\[ X \text{ space, } A = (C^*X, d), \ W \in H^e(X)(\text{cocycle}) \]

"Curved algebra \( (A,W) \)
\[ \text{cat. of modules (O10v)} \]
\[ "A/W" = A[\Theta], \ d\Theta = W. \]

\[ \text{A-perfect } A/W \text{ complexes} \quad ("D \text{Sing } W^-(0)") \]
\[ \text{A}/W\text{-perfect complexes} \]
Matrix factorization description

\[ CH_* (A, W) = CH (A^+), b + [W, \cdot] \]

before, \( C^* (LX) \), \( TW: \) transgression of \( W \) to \( H^{odd} (LX) \)

now \( CW C^* (LX) : S + TW \) (new deg \( W > 2 \))

before, had an equivalence

\[ C^* (X) - \text{mod} \cong C^* (\Omega X) - \text{mod} \]

now, \( (C^* (X), W) \cong TW C^* (\Omega X) \) "twisted Koszul duality"

(Also said: before taming, had an embedding of LHS \( \rightarrow \) RHS. Tame now also, LHS might be a bit large of right).

less info

When \( X = BG \),

\[ C^* (X): (\text{Sym} g^*)^G - \text{mod}, \quad "\text{good}" \]

\[ C^* (G): (\Lambda g)^G - \text{mod}, \quad "\text{bad}" \quad \text{(esp. of normal case).} \]

\[ \mathbb{X} - \text{ringed space (sheaf of ring spectra)} \]

sheaf of rings

\[ \mathcal{U} \rightarrow K^* (\mathcal{U}) \] (really want \( K \)-spectra)

\[ W \rightarrow H^4 (X) (\approx K^* (X)) \]

\[ H^1 (\mathbb{X}; K (\mathbb{Z}, 3)) \]

\[ \mathcal{U}_i \cap \mathcal{U}_j \]

\[ K^* (\mathbb{X}) - \text{mod by patching} \]

\[ \mathcal{U}_i \rightarrow K^* (\mathcal{U}_i ) - \text{module } M_i \]

\[ M_i \otimes_{K^* (\mathcal{U}_i )} K^* (\mathcal{U}_i \cap \mathcal{U}_j) \cong M_j \otimes_{K^* (\mathcal{U}_j )} K^* (\mathcal{U}_i \cap \mathcal{U}_j) \]
On each \( U_i \cap U_j \), have twisted \( \mathfrak{m}_j \mathfrak{k}^*(U_i \cap U_j) \) (sections of a line bundle...)

Define curved \( \mathfrak{k}^*(X) \)-mod cat. by

\[
M_i \otimes_{\mathfrak{k}^*(U_i)} \mathfrak{w}_j \mathfrak{k}^*(U_i \cap U_j) \cong M_j \otimes_{\mathfrak{k}^*(U_j)} \mathfrak{k}^*(U_i \cap U_j).
\]

\[
H^4(X; \mathbb{Z}) = H^2(X; \mathfrak{k}(\mathbb{Z}, 2)) \cong \text{GL}_1(\mathfrak{k}^*).
\]

If \( X = BG \),

\[
W \in H^4(BG; \mathbb{Z}) \quad \text{quad. fan. on } G,
\]

multi-valued quad. fan on \( G/G \) (space of conj. classes),

potential \( dw : T \rightarrow T^* \) \( \Rightarrow \) \( W \) curved \( \mathfrak{k}^* \)-module cat.

\( \ker dw \) defined, Hess defined.

(Ref: paper by Teleman-Woodward.)