Let $\Sigma$ be a closed surface of genus $g \geq 2$.

$$\text{Hom}(\pi_1 \Sigma, \text{SO}(3))/\text{SO}(3)$$

has 2 connected components, corresponding to flat $\nabla$ on the trivial resp. non-trivial bundles $\Sigma$.

These guys are singular, which is problematic.

$$M(\Sigma) = \text{Hom}(\pi_1 \Sigma, \text{SU}_2)/\text{SU}_2$$

$$= \left\{ (A_1, B_1, \ldots, A_g, B_g) \in \text{SU}(2)^g \mid \prod_{i=1}^{g} [A_i, B_i] = -I \right\}/\text{SU}_2$$

the "twisted" representation space.

$$M(\Sigma) \rightarrow \text{Hom}(\pi_1 \Sigma, \text{SO}(3))/\text{SO}(3) \text{ is a } \left(\mathbb{Z}/2\right)^g \cong H^1(\Sigma, \mathbb{Z}/2)$$

branched over the "non-trivial" component. (i.e. non-trivial bundle part)

$M(\Sigma)$ is a smooth, closed $(6g-6)$-dimensional symplectic manifold, 1-connected.

(Emb: Spaces of 1-forms on surfaces tend to be symplectic, natural skew-symmetric pairing)

Viewed in terms of reps of $\pi_1(\Sigma \setminus p)$, then $\pi_0 \text{Diff}_+(\Sigma \setminus p)$ acts on $M(\Sigma)$. Actually, this factors through a rep of $\tilde{\pi}_1$

$$1 \rightarrow H^1(\Sigma; \mathbb{Z}/2) \rightarrow \tilde{\pi}_1 \rightarrow \tilde{\pi}_0 ^+ \rightarrow 1$$

$$\tilde{\pi}_1 \rightarrow \text{Symp}(M(\Sigma))$$

Classical: this injective.

Conjecture: $\tilde{\pi}_1 \rightarrow \pi_0 \text{Symp}(M(\Sigma))$ is injective.

ASM: Explain related circle of ideas yielding proof when $g=2$.

Work-in-progress.

Remarks: Classical surgery theory $\Rightarrow \tilde{\pi}_1 \rightarrow \pi_0 \text{Diff}(M(\Sigma))$ is not faithfual.

- $\tilde{\pi}_1 \rightarrow \pi_0 \text{Symp} M(\Sigma)$ is non-trivial on Torelli gp. (not detected by action on $H^* M(\Sigma)$)
Let \( f : \Gamma_2 \to \Gamma_2 \) have a pseudo-Anosov component. Then, \( HF(pff)^k \) grows exponentially in rank with \( k \).

**Thurston:** 3 classification of surface diffeomorphisms -
- periodic
- reducible (preserve a collection of simple closed curves up to isotopy)
- pseudo-Anosov (chaotic maps, exponential growth of periodic points).

*Under iteration*

Suppose key result is known for some \( g \). Suppose \( p(f) = \pi d \in \pi_0 \text{Symp } \Sigma \).

**Easy:** \( f \sim \text{id} \in \Gamma_g \).
- \( f \) can't be periodic; action of \( p(f) \) on \( H^* \Sigma \) remembers action of \( f \) on \( H^* \Sigma \)
- key claim \( \Rightarrow \) \( f \) can't be pseudo-Anosov.
- \( f \) reducible, built out of Dehn twists on disjoint separating curves.
  - e.g. \( f = t_{\sigma}, \) or separating. But \( f \circ t_{\sigma} \circ t_{\sigma} \circ \) is \( \not\text{Anosov} \).
  If \( p(f) = \text{id}, \) \( p(\sigma \circ t_{\sigma}) = p(\sigma) \) \( \Rightarrow \) \( HF(t_{\sigma}) \) grows exponentially in rank, but this is nonsense (not pointlessly), b/c can explicitly see that it grows linearly in rank.
  (explicitly use the fact that \( t_{\sigma} \) is a fibered Dehn twist).

**Stranger conjecture:**
\( HF(f) \) is a summand of \( HF(\rho(f)) \).

**Motivation**
- Munoz: \( QH^*/\Sigma \) \( \cong \bigoplus_{i=-g+1}^{g-1} H^*(\text{Sym}^{g-1-i} \Sigma) \)
- \( HF(\rho(\text{id})) \)

\( \otimes \) this is the eigenvalue splitting with respect to \( \rho_\Sigma: \Sigma \to \Sigma \):

**Question:** Is \( HF(p(f)) \approx \bigoplus_{i=-g+1}^{g-1} HF(f^{(i)}) \) \( \forall f, \) \( f^{(i)} \) act on \( \text{Sym}^{g-1-i} \Sigma \)?

(intentionally vague w.r.t. coefficients... certainly \( f \) won't change \( 0 \), maybe can use \( C \) if you make careful choices, otherwise \( \Lambda \)).
\[ H^F(p(f)) \xrightarrow{\text{Dist.}} H^F_{\text{inst.}}(Y_p) \]

\[ \text{fibred 3-manfold defined by } f \]

\[ H^F_{\text{inst.}}(f^{(1)}) \text{ in some way.} \]

**Guess:** \( H^F(p(f)) \) splits into e-spaces according to \( \text{Spec}(x^c_i), \)

\[ \text{Top e-space } \leftrightarrow \text{Top } \xrightarrow{\text{HF}} \text{H}^F(f^{(g-1)}) \text{ is rank 1 on } \text{Sym}(\Sigma) \]

\[ \text{Next e-space } \leftrightarrow \text{H}^F(f^{(g-2)}), \text{i.e. } \text{HF}(f) \text{ is a summand of } \text{HF}(p(f)) \]

**Proposition:** \( \text{HF}(p(f)) \) has a distinguished \( \text{rk 1 } \) summand (corresponding to \( x^c_i \), acting by \( 4(g-2) \)).

\[ \text{Rank: } \]

\[ \text{HF}(p(f)) \text{ never vanishes } \]

\[ Y \to S' \text{ fibred 3-fold, } E \text{ non-trivial } S^3 \text{-rep}'n \text{ of } \pi_1 Y \]

\[ \text{If } f \in \text{Torelli grp, } p(f) \xrightarrow{\text{smoothly}} \text{fixed-pt-free diffeomorphism } \quad (\mathcal{M}(\Sigma)_1 = 0). \]

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**2nd Motivation:** Homological Mirror Symmetry

**Fact:** \( \mathcal{M}(\Sigma)_g \xrightarrow{\text{sympl.}} \text{Space of } P^{g-2} \text{ in } Q_0 \cap Q_1 \leq P^{2g+1} \)

\( \text{(Due to Ramras and Neeman, } g = 2, M(\Sigma)_g = Q_0 \cap Q_1 \leq P^5 \text{ is a Fano variety, (All } M(\Sigma)_g \text{ are)} \)

If \( X \) is Fano, \( X \) has a collection of Fukaya categories

\[ \{ \mathcal{F}(X, \lambda) \}_{\lambda \in L_X} \text{, non-zero only if } \lambda \in \text{Spec}(x^c_1 : QH^*(X)) \]

If \( L \subseteq X \) is a Lagrangian submanifold, define \( \text{HF}(L, L) \) if \( L \) is weakly unobstructed, meaning \( \mu_0(L) = \gamma[L] \), in this case \( L \) belongs to \( \mathcal{F}(X, \lambda) \)

In \( \text{CF}(L_1, L_2), d^2 = \mu_0(L_1) - \mu_0(L_2). \)

**Hod:** \( X \) Fano, mirror \( LG \) model \( \xrightarrow{\text{def.}} Y \rightarrow C \)

\[ D^b(X) \leftrightarrow D^b_{\text{Fuk}}(W) \]

\[ D^b(X) \leftrightarrow D^b_{\text{Sag}}(Y, W) \]

\[ \otimes D^\bullet_{\text{Fuk}}(X, \lambda) \leftrightarrow \otimes_{\text{crit}(W)} D^b_{\text{Sag}}(Y, \lambda) \]
If we blow up $X$ along $\mathcal{B}$, $\mathbb{D}^b(\mathcal{B}) \hookrightarrow \mathbb{D}^b(\mathbb{B} \cup \mathcal{X})$ (semi-orthogonal decomposition).

In some cases, understood by adding singular fibres to $W: X \to \mathcal{C}$

(point: adding singular fibres just adds summands to $\mathbb{D}^b(\mathcal{X})$, maybe also true for $\mathbb{D}^n_F(X, \lambda)$?)

Conjecture:

$$\mathbb{D}^n_{F}(\Sigma g) \cong \mathbb{D}^n_{F}(Q_0 \cdot Q_1; \emptyset)$$

in $\mathbb{P}^{2g+1}$

$$\mathbb{D}^n_{F}(\Sigma g; \emptyset(2g-2))$$

as $\mathbb{Z}/2$-graded categories over $K$-alg. closed, char. 0.

(2nd highest term, in progress)

2nd highest term, in progress.

Results on derived categories:

(i) Bondal-Orlov:

$$\Sigma \to \mathbb{P}^{2g+1} \cong \{ z_0, \ldots, z_{2g+1}\}, \quad Q_0 = \sum z_i^2 = 0, \quad Q_1 = \sum \lambda_i z_i^2 = 0 \leq \mathbb{P}^{2g+1},$$

hyper elliptic curve

branch pts.

then $\mathbb{D}^b(\Sigma) \hookrightarrow \mathbb{D}^b(Q_0 \cdot Q_1)$ (mirror of the statement we're trying to prove).

$[\Sigma]$ is a moduli space of sheaves on $Q_0 \cdot Q_1$, universal bundle on $\Sigma, \times (Q_0 \cdot Q_1)$, gives a Fourier-Mukai.

(ii) Kapranov-Kurstenov:

$$\mathbb{D}^b(\text{anodic} \in \mathbb{P}^{2g+1}) \hookrightarrow \mathbb{D}^b(\text{2pts.}) \hookrightarrow \mathbb{D}^b(\text{base}) \hookrightarrow \mathbb{D}^b(\text{upper})$$

(semi-orthogonal)

$$\mathbb{P}^{2g+1} \cong Q_0 \cdot Q_1 \cong Q_0 \cdot Q_1 \cap Q_2 \cong \cdots$$ (complete intersections of quadrics).

$$\varphi \leq 2 \text{pts.} \leq \Sigma g \leq \mathbb{Z} \cong \mathbb{P}^{2g+1} \cong \mathbb{D}^b(\text{upper}) \hookrightarrow \mathbb{D}^b(\text{base})$$

(2:1 cover of $\mathbb{P}^g$ over $\mathbb{D}^{2g+1}$ where above is an $\mathbb{A}^1$-linear system.

Inductively obtained by $2 \text{pts.} \hookrightarrow Q_0$, so this should motivate the strategy here)

Proposition: Let $Q^g \leq Q^{2g+1}$ be a quadric. $Q$ contains a unique log sphere which is the v. cycle of an alg. degenerating, call $L$, $L$ split generates $F(Q, 0)$

$H^2(L, L) \cong \mathbb{K}[t]/(t^2 = 1)$ is semi-simple. (Arason formal)

$\Rightarrow \mathbb{D}^n_F(Q, 0) \cong \mathbb{D}^n_F(S^0)$ as $\mathbb{Z}/2$-graded categories.
Proposition: $D^\infty F(Q\cap Q_1;0) \to D^\infty F(Z;\mathbb{I})$ if $Z = Bl_{Q\cap Q_1}(\mathbb{P}^{2g+1})$.

Note: $Z \to \mathbb{P}^1$ is a Lefschetz fibration over $\mathbb{P}^1$; $2g+2$ singular fibers at $\mathbb{I}$; $\Sigma \to \mathbb{P}^1$ is a $2$-cover, $2g+2$ branch points.

These have the same monodromy, via natural rep'n of

$$\pi_1(\mathbb{P}^1 \setminus \mathbb{I}) \to \mathbb{Z}/2(\Sigma)$$

$$\to \mathbb{Z} \to \mathbb{Z}/2$$

Rohn twist is $L$ for $Z$.

Upshot:

$$\Sigma \leftrightarrow Q \cap Q_1 \quad \text{Lef. fibration theory} \quad D^\infty F(Z_{\neq 0}) \approx D^\infty F(\Sigma_{\neq 0})$$

(we want deformations of these, though deformations spaces are so-dim $1$, so can't do something naive) (the mirror result is true, though).

$$B \times \mathbb{P}^1$$

Remarks on proof: $B = \{ H \neq 0 \} \times \mathbb{P}^1$, $E$ $Bl_B \geq E''$

$B \times (B \times \mathbb{P}^1)$

$\{ H \neq 0 \}$

Log'g: $\Delta B \times S^1_e$

Correspondences

$E \times Bl_B \times$

$D\{ V_e \} \quad \text{boundaries of}

key idea: use these to get

fully faithful embedding

(by work of Mau-Werhein-Woodward)

need to choose the radius correctly; it's $\kappa/2$, a blow-up param.)